Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2015.w05si

Vol. **8**, No. 1, pp. 47-77
February 2015

Selected Recent Applications of Sparse Grids

Benjamin Peherstorfer[1,*], Christoph Kowitz[2], Dirk Pflüger[3] and Hans-Joachim Bungartz[1]

[1] *Scientific Computing Group, Dept. of Computer Science, Boltzmannstr. 3, 85748 Garching, Germany.*
[2] *Institute for Advanced Study, Technische Universität München, Lichtenbergstr. 2a, 85748 Garching, Germany.*
[3] *Institute for Parallel and Distributed Systems, University of Stuttgart, Universitätsstr. 38, 70569 Stuttgart, Germany.*

Received 2 November 2013; Accepted 23 July 2014

Abstract. Sparse grids have become a versatile tool for a vast range of applications reaching from interpolation and numerical quadrature to data-driven problems and uncertainty quantification. We review four selected real-world applications of sparse grids: financial product pricing with the Black-Scholes model, interactive exploration of simulation data with sparse-grid-based surrogate models, analysis of simulation data through sparse grid data mining methods, and stability investigations of plasma turbulence simulations.

AMS subject classifications: 65M60, 65M22, 62P05, 65D05, 62H30, 65F15

Key words: Sparse grids, hierarchical decomposition, adaptivity, combination technique.

1. Introduction

The underlying principle of sparse grids is a hierarchical basis that leads to a hierarchical decomposition of function spaces into hierarchical increments. These hierarchical increments are then the starting point for optimization problems with which one constructs approximation spaces for function spaces by selecting only those increments which have a sufficiently good cost-benefit ratio; the costs equal the dimension of the approximation space, and the benefit is related to the interpolation error in a given norm. Sparse grid spaces are optimal approximation spaces with respect to these criteria for the space H^2_{mix}, which contains functions with bounded, mixed derivatives up to order two. The corresponding theory is presented in the survey article [9].

Sparse grids have been applied to a variety of computational tasks. The purpose of this article is to highlight four selected and recently presented real-world applications.

*Corresponding author. *Email address:* `peehersto@in.tum.de` (B. Peherstorfer)

We summarize a few key facts on sparse grids in Section 2 and clarify the notation. Our presentation particularly emphasizes that sparse grids build on hierarchical and multi-level principles. In Section 3, we consider financial product pricing, where the multi-dimensional Black-Scholes equation is solved. In Section 4, a sparse-grid-based surrogate modeling approach for interactive visual exploration of parametrized simulation data is discussed. We construct a surrogate model for a building information model (BIM) that simulates a flow through a building. We continue in Section 5 with a data-driven problem where we analyze simulation data with sparse grid data mining methods. Finally, multi-dimensional eigenvalue problems for plasma turbulence simulations are solved on sparse grids in Section 6. The eigenvalues and eigenvectors give information about whether the plasma is stable or not.

2. Sparse grid spaces

We give a brief overview of sparse grids and particularly emphasize the strong relationship to hierarchical and multi-level computational methods. We also discuss the combination technique, spatial adaptivity, and list a few software libraries that implement common sparse grid routines. We do not go into the details of sparse grid theory but only present the preliminaries for the following applications. We refer to the survey article [9] for details.

2.1. Full grid spaces and their hierarchical decomposition

Let $\mathcal{V}$ be a function space with domain $\Omega = [0,1]$ and homogeneous boundaries, e.g., $\mathcal{V} = H_0^2(\Omega)$. We discretize a function $f \in \mathcal{V}$ by constructing its interpolant in the finite-dimensional space $\mathcal{V}_\ell^{(\infty)} \subset \mathcal{V}$ of piecewise linear functions with mesh width $2^{-\ell}$. The accuracy of the interpolant is controlled by the level ℓ of the space. The space $\mathcal{V}_\ell^{(\infty)}$ is spanned by the basis functions

$$\varphi_i(x) := \phi(2^\ell x - i)\,, \quad 1 \leq i < 2^\ell\,, \tag{2.1}$$

where $\phi : [-1,1] \to \mathbb{R}$ with $\phi(x) = \max\{1 - |x|, 0\}$. The interpolant $\hat{f} \in \mathcal{V}_\ell^{(\infty)}$ of $f \in \mathcal{V}$ can be represented as a linear combination

$$\hat{f} = \sum_{i=1}^{N} a_i \varphi_i$$

of the basis functions (2.1) and coefficients $\boldsymbol{a} = [a_1, \cdots, a_N]^T$ where $N = 2^\ell - 1$. It follows that $a_i = f(i \cdot 2^{-\ell})$ for $i = 1, \cdots, N$. On the one hand this means that $\hat{f}$ is easy (w.r.t. the implementation effort) and cheap (w.r.t. the computational costs) to compute. On the other hand, the coefficients $\boldsymbol{a}$ do not lead to an ordering of the basis functions with respect to the benefit of including a basis function into the linear