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Algebraic Theory of Two-Grid Methods

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Abstract. About thirty years ago, Achi Brandt wrote a seminal paper providing a convergence theory for algebraic multigrid methods [Appl. Math. Comput., 19 (1986), pp. 23–56]. Since then, this theory has been improved and extended in a number of ways, and these results have been used in many works to analyze algebraic multigrid methods and guide their developments. This paper makes a concise exposition of the state of the art. Results for symmetric and nonsymmetric matrices are presented in a unified way, highlighting the influence of the smoothing scheme on the convergence estimates. Attention is also paid to sharp eigenvalue bounds for the case where one uses a single smoothing step, allowing straightforward application to deflation-based preconditioners and two-level domain decomposition methods. Some new results are introduced whenever needed to complete the picture, and the material is self-contained thanks to a collection of new proofs, often shorter than the original ones.

AMS subject classifications: 65F08, 65F10, 65F50

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1. Introduction

Multigrid methods are among the most efficient iterative techniques to solve large sparse systems of linear equations. These methods combine two different iterations: a smoothing iteration, which is often a simple iterative method like the Gauss–Seidel method, and a coarse grid correction, which consists in computing an approximate solution to the residual equation on a coarser grid with fewer unknowns.

Perhaps because of this combination two different processes, multigrid methods are difficult to analyze. Abstract theories (e.g., [21,46]) are restricted to discretized partial differential equations on a regularly refined grid, and allow one to obtain only qualitative results. Fourier and local mode analyses (e.g., [55]) yield sharp quantitative estimates, but only when the system to solve stems from a constant or smoothly variable

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grid stencil. In all other cases, the algebraic theory [6,10,17,18,42,48,51,57] appears as the right tool, despite its own limitations. In particular, it allows the assessment of "algebraic" multigrid (AMG) methods [8], in which the coarse grid correction is not related to the discretization on a coarser grid, but defined by applying proper algorithms to the system matrix. Examples of works where the theoretical results have helped to the design and/or the analysis of AMG methods include [2, 10, 14, 38, 39, 43, 48]; the algebraic theory also gave theoretical foundations to the coarsening by compatible relaxation [7,24], see [9,17,42].

Early analyses of multigrid methods using essentially algebraic arguments trace back to the eighties and include [3, 20, 26, 31–33]. They were quickly followed by Brandt's seminal paper [6], which provides the first convergence theory applicable to (and intended for) AMG methods. Since then, the theory has been improved and extended in a number of works; see, e.g., [10, 17, 18, 42, 48, 51, 57]. Now, each reference highlights its own improvements, and the reader searching for a clear summary or overview of the state of the art has to go through quite many specialized works. Moreover, many of these works are restricted to a specific smoothing scheme, making difficult the setting up of a clear picture of the available results.

Our main goal in this paper is to present such a clear picture, highlighting in particular the differences induced by the type of smoothing scheme.

The organization of the paper is a bit unusual. After the introduction of the general setting and the statement of the common assumptions (§2), we state in §3 the set of results as "Facts", which are given without proof nor comment. All comments are gathered in §4, whereas proofs are delayed until §5. Most of the facts are indeed not new, and hence have already been proved and commented in the original references. Regarding comments, we therefore focus on those which compare the facts that are similar but cover different situations, a viewpoint seldom taken in the literature. Logically, this can be done only after all facts have been stated.

Regarding proofs, some readers may feel that they are unnecessary. However, for most facts we are able to give a new proof, generally shorter than those in the original references, and we also often condense the proof of several facts in a single one. This allows us to be self-contained while keeping §5 to a fairly reasonable size. Moreover, some of the results are new and require in any case a proof. For instance, the necessary and sufficient conditions for the nonsingularity of the two-grid preconditioner are seemingly stated for the first time (Fact 1.1), and we are not aware of previous studies considering in detail the effect of the number of smoothing steps on the theoretical estimates (Fact 5.4).

The chosen format (inspired by [22]) does not allow us to include a thorough discussion of past contributions and how they influenced the progress in the field. For each fact stated in §3 we mention its origin in parenthesis, and we further make some connection with the literature in §4, especially in Remark 4 devoted to historical comments. However, this yet gives only a very partial account of previous developments. In particular, one should not forget that more general or sharper bounds are always indebted to their predecessors: even when the new proof is very different in nature,