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A FEM-Multigrid Scheme for Elliptic Nash-Equilibrium Multiobjective Optimal Control Problems

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Abstract. A finite-element multigrid scheme for elliptic Nash-equilibrium multiobjective optimal control problems with control constraints is investigated. The multigrid computational framework implements a nonlinear multigrid strategy with collective smoothing for solving the multiobjective optimality system discretized with finite elements. Error estimates for the optimal solution and two-grid local Fourier analysis of the multigrid scheme are presented. Results of numerical experiments are presented to demonstrate the effectiveness of the proposed framework.

AMS subject classifications: 15A12; 65F10; 65F15
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1. Introduction

The simultaneous optimization of several objectives depending on different control mechanisms and subject to a unique model constraint defines a special class of multiobjective optimization problems. In the case of finite-dimensional models, a representative class of multiobjective optimization problems are game theory problems. In the infinite-dimensional case, multiobjective optimization usually refers to differential models. Specific infinite-dimensional application problems are aerodynamic shape optimization with the purpose of lift maximization and drag minimization [9, 11], and multi-loading structural design [17].

The pioneering results on finite-dimensional multiobjective optimization can be found in the works of Pareto, Nash, and Stackelberg; see e.g. [7, 8, 10]. On the other hand, the literature on infinite-dimensional multiobjective problems is much more recent and sparse. In particular, the Pareto solution concept may be found in, e.g., [12, 13], while much less is known regarding the Nash solution concept in multiobjective optimization problems governed by differential systems; see the very recent papers [1, 14, 15, 18]. Furthermore, it is only recently that efficient methods for

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the solution of finite-dimensional Nash equilibrium problems have been investigated, whereas in the infinite-dimensional case the development and analysis of solution techniques is at its infancy. The purpose of this paper is to contribute to this development with a finite-element multigrid solution strategy for infinite-dimensional elliptic Nash-equilibrium multiobjective optimization problems. Specifically, we focus on an elliptic partial differential equation (PDE) with distributed controls and two objectives corresponding to two controls that are sought to obtain a Nash equilibrium. This setting is similar to that in [1, 14, 15], where finite-dimensional solution techniques are used to solve a finite-differences discretized multiobjective optimization problem. In contrast, we consider a FEM discretization and a multigrid solution strategy that is appropriate to solve elliptic PDE-optimization problems with optimal complexity at any mesh size. We remark that the present work addresses many novel issues. In fact, it considers a finite-element discretization of a Nash-equilibrium multiobjective optimality system in domains that are difficult to handle with finite differences and provides accuracy results for this setting. Further, it develops a powerful multigrid strategy that accommodates constraints on the control and is supported by local Fourier analysis results. However, while this paper provides a novel ground-making contribution to many different aspects of the numerical solution to elliptic Nash-equilibrium multiobjective optimization problems, many issue remain subject of further investigation. Therefore, we limit our theoretical analysis of the discretization to a one-dimensional setting and analyses the multigrid convergence only in the unconstrained case.

This paper is organized as follows. In Section 2, we formulate a class of elliptic Nash-equilibrium multiobjective optimization problems. We consider two objectives and two constrained controls that are defined in two subsets of the computational domain. Correspondingly, we introduce the optimality system that characterizes the solution to our multiobjective optimization problem. In Section 3, we illustrate a finite-element discretization of the optimality system and investigate the resulting approximation properties. We extend the analysis of [16] and prove $3/2$-order convergence for the control functions. In Section 4, we discuss a multigrid method for the solution of the optimality system. We present a full multigrid scheme with collective smoothing that is able to accommodate the presence of inequality constraints. The optimal convergence properties of our multigrid scheme are investigated in Section 5 using local Fourier analysis. In Section 6, we present results of numerical experiments to validate our Nash-equilibrium optimal control framework. In particular, we validate the theoretical estimates and the computational performance of the proposed FEM-multigrid scheme. Our FEM multigrid scheme is found to be efficient and robust for a large range of the weights of the costs of the controls and provides second-order optimal solutions. A section of conclusions completes this work.

2. An elliptic multiobjective optimal control problem

In this section, we formulate a class of elliptic multiobjective Nash-equilibrium distributed control problems as follows.