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Local Multilevel Method on Adaptively Refined Meshes for Elliptic Problems with Smooth Complex Coefficients

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Abstract. In this paper, a local multilevel algorithm is investigated for solving linear systems arising from adaptive finite element approximations of second order elliptic problems with smooth complex coefficients. It is shown that the abstract theory for local multilevel algorithm can also be applied to elliptic problems whose dominant coefficient is complex valued. Assuming that the coarsest mesh size is sufficiently small, we prove that this algorithm with Gauss-Seidel smoother is convergent and optimal on the adaptively refined meshes generated by the newest vertex bisection algorithm. Numerical experiments are reported to confirm the theoretical analysis.

AMS subject classifications: 65F10; 65N30

Key words: Local multilevel algorithm, Adaptive finite element method, Complex coefficients, Gauss-Seidel smoother.

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1. Introduction

Multilevel or multigrid methods are very efficient for solving the systems arising from the approximation of elliptic boundary value problems. The convergence properties of multigrid methods on quasi-uniform meshes have been studied by [2, 5, 7, 9, 11, 16, 18, 30, 31] and the references therein. Multigrid methods on locally refined meshes was first introduced by Brandt [10]. Subsequently, the techniques on locally refined meshes have been widely investigated, such as the multilevel adaptive technique (MLAT) in [1, 20, 27] and the fast adaptive composite grid (FAC) methods in [21–24]. Recently, Wu and Chen [33] proposed a multigrid V-cycle algorithm with Gauss-Seidel smoother on adaptive refined meshes generated by the newest vertex bisection algorithm, they proved that this algorithm is convergent and optimal when the smoother performing only on the new nodes and their "immediate" neighbors (i.e., the old nodes

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whose support or the nodal basis function has changed). Therefore, this multigrid algorithm is more robustly applied to practical problems. Based on the well known Schwarz theory [29], Xu and Chen [13, 32] proved the convergence of some local multilevel algorithms with Gauss-Seidel and Jacobi smoothers applied to second order elliptic boundary value problems.

The purpose of this paper is to study local multilevel algorithm applied to second order complex coefficient elliptic problems. Complex coefficient problems arising from time harmonic scattering and radiation have been studied in [4,8,14,19]. Gopalakrishnan and Pasciak [17] proved that multigrid algorithm is convergent on quasi-uniform meshes for second order elliptic problems with smooth complex coefficients. In this paper, adopting the techniques developed in [13], we analyze the convergence of local multilevel algorithm with Gauss-Seidel smoother applied to complex coefficient elliptic problems. The key is how to extend the assumption conditions presented in [13, 30] to complex coefficient case and verify them. Based on the perturbation arguments proposed in [17], it is shown that this algorithm converges if the mesh size of the coarsest grid is sufficiently small.

We shall use the standard notation for the Sobolev spaces $W_p^m(\Omega)$ with norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$ [15]. For $p=2$, we denote $H^m(\Omega)=W_2^m(\Omega)$, $H_0^1(\Omega)=\{v\in H^1(\Omega): v\,|_{\partial\Omega}=0\}$, $\|\cdot\|_m=\|\cdot\|_{m,2}$ and $\|\cdot\|=\|\cdot\|_{0,2}$. Throughout this paper, the notation $x\lesssim y$ (or $x\gtrsim y$) represent the inequality $x\leq Cy$ (or $x\geq Cy$), where the positive constant C is independent of all the variables in the inequality. The notation $x\approx y$ is equivalent to the statement that $x\lesssim y$ and $x\gtrsim y$.

The rest of the paper is organized as follows. In Section 2, we describe the multilevel method on adaptively refined meshes applied to a model of elliptic problem with complex coefficient. In Section 3, we present a convergence estimate for the local multilevel algorithm with Gauss-Seidel smoother. Finally, in Section 4, some numerical examples are given to show the efficiency of the local multilevel algorithm.

2. Multilevel algorithm on adaptively refined meshes

To illustrate the main idea, we consider the following second order elliptic problem [17]

$$\begin{cases} -\nabla\cdot(\alpha(x)\nabla u)=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a polygonal domain in R^2 and $\alpha(x):\Omega\rightarrow C$ is a complex valued non-vanishing function in $C^2(\overline{\Omega})$, $f\in L^2(\Omega)$. Assuming that there exists a positive constant α_0 such that $|\alpha(x)|\geq\alpha_0$ for all $x\in\Omega$.

For brevity, we omit the variable x in the following exposition. The variational form of (2.1) is to find $u\in H_0^1(\Omega)$ such that

$$A(u,v)=(f,v)\quad\forall\, v\in H_0^1(\Omega), \tag{2.2}$$