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A Staggered Discontinuous Galerkin Method with Local TV Regularization for the Burgers Equation

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Abstract. The staggered discontinuous Galerkin (SDG) method has been recently developed for the numerical approximation of partial differential equations. An important advantage of such methodology is that the numerical solution automatically satisfies some conservation properties which are also satisfied by the exact solution. In this paper, we will consider the numerical approximation of the inviscid Burgers equation by the SDG method. For smooth solutions, we prove that our SDG method has the properties of mass and energy conservation. It is well-known that extra care has to be taken at locations of shocks and discontinuities. In this respect, we propose a local total variation (TV) regularization technique to suppress the oscillations in the numerical solution. This TV regularization is only performed locally where oscillation is detected, and is thus very efficient. Therefore, the resulting scheme will preserve the mass and energy away from the shocks and the numerical solution is regularized locally near shocks. Detailed description of the method and numerical results are presented.

AMS subject classifications: 65M10, 78A48

Key words: discontinuous Galerkin method, TV regularization, energy conservation, mass conservation, staggered grid.

1. Introduction

The staggered discontinuous Galerkin (SDG) method is a new class of discontinuous Galerkin methods that are defined on some easy-to-construct staggered grids, which can be either structured or unstructured. The SDG method is first developed by Chung and Engquist [7,8] for the acoustic wave propagation, and is then developed for a large class of partial differential equations. In particular, SDG methods are developed for scalar elliptic problems in Chung, Kim and Widlund [10], the convection-diffusion

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equation in Chung and Lee [12], the two-dimensional curl-curl problem in Chung and Lee [11], time-harmonic problems in Chung and Ciarlet [3], three-dimensional time-dependent Maxwell's equations in Chung, Ciarlet and Yu [4], wave simulations in heterogeneous media for geophysical applications in Chung, Efendiev, Gao and Gibson [9,14] and the Stokes system in Kim, Chung and Lee [19]. The method relies on the careful design of a pair of finite element spaces that satisfy some discrete inf-sup conditions and, more importantly, some staggered continuity conditions. More precisely, the functions in these two finite element spaces are continuous at different element boundaries. Moreover, they are globally discontinuous and are only continuous locally in a union of a few neighboring elements. The main consequence is that no numerical flux and penalty parameters are required in contrast to the non-staggered DG methods. An important advantage of such methodology is that the numerical solution automatically satisfies some conservation properties which are also satisfied by the exact solution. For example, the method provides energy conservation for wave propagation [4,7,8], provides mass and energy conservation for convection-diffusion equations [12], and provides divergence free velocity for Stokes flows [19]. In respect of wave propagation, the method also gives smaller dispersion error [2,4]. Historically, staggered grid methods have also been widely used in the context of finite difference and finite volume methods. For instances, they are applied to wave propagation problems in [5,6] and fluid flow problems in [1,22].

In this paper, we will develop a SDG method for the Burgers equation. It is also our first step in the development of SDG methods for the more general non-linear conservation laws. In particular, we consider the Cauchy problem of the following equation

$$u_t(t,x)+u(t,x)\,u_x(t,x)=0 \tag{1.1}$$

in some interval $I=[a,b]$, subject to the initial condition

$$u(0,x)=u_0(x), \qquad x\in I.$$

Numerical methods for hyperbolic conservation laws are widely studied. For example, in Harten, Engquist, Osher and Chakravarthy [16] and Shu [23], the Essentially Non-Oscillatory (ENO) schemes and the Weighted Essentially Non-Oscillatory (WENO) schemes are developed with great success. In addition, to avoid the time-consuming Riemann solvers, some relaxation schemes for solving hyperbolic systems of conservation laws are proposed and analyzed in Jin and Xin [18]. For an extensive discussion of numerical methods, see LeVeque [20,21]. Nevertheless, the issue of energy conservation is usually not the focus of these methods. Recently, Jameson [17] proposes some finite difference schemes that give energy conservation for nonlinear hyperbolic conservation laws, and addresses the importance of energy preserving schemes. This motivates us to design DG schemes that also give energy conservation. One advantage of the DG idea is that it allows an easy construction of higher order approximations [13]. In fact, the lowest order version of our SDG method coincides the scheme developed in Jameson [17], and thus our SDG scheme can be regarded as a generalization of the