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Superoptimal Preconditioners for Functions of Matrices

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Abstract. For any given matrix $A \in \mathbb{C}^{n\times n}$, a preconditioner $t_U(A)$ called the superoptimal preconditioner was proposed in 1992 by Tyrtyshnikov. It has been shown that $t_U(A)$ is an efficient preconditioner for solving various structured systems, for instance, Toeplitz-like systems. In this paper, we construct the superoptimal preconditioners for different functions of matrices. Let f be a function of matrices from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$. For any $A \in \mathbb{C}^{n\times n}$, one may construct two superoptimal preconditioners for $f(A)$: $t_U(f(A))$ and $f(t_U(A))$. We establish basic properties of $t_U(f(A))$ and $f(t_U(A))$ for different functions of matrices. Some numerical tests demonstrate that the proposed preconditioners are very efficient for solving the system $f(A)\mathbf{x} = \mathbf{b}$.

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1. Introduction

Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix. Define the set

$$\mathcal{M}_U \equiv \Big\{ U^*\Lambda U \mid \Lambda \text{ is any } n\times n \text{ diagonal matrix} \Big\},$$

where U^* means the conjugate transpose of U. For any $A \in \mathbb{C}^{n\times n}$, the superoptimal preconditioner $t_U(A)$ is defined to be the minimizer of

$$\min \|I_n - W^{-1}A\|_F$$

over all nonsingular matrices $W \in \mathcal{M}_U$, where $\|\cdot\|_F$ denotes the Frobenius matrix norm.

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The superoptimal preconditioner was proposed by Tyrtyshnikov [20] in 1992. It was also studied by Chan *et al.* [5, 6]. The superoptimal preconditioner was used to solve some ill-conditioned problems appeared in image deblurring [2] and other structured systems [4, 7, 11, 13]. For more useful properties of $t_U(A)$, we refer to [9, 10, 14].

Recently, Jin *et al.* [16] study the optimal preconditioners for matrix functions. For any given $A \in \mathbb{C}^{n\times n}$, the optimal preconditioner $c_U(A)$ is defined in [8] to be the minimizer of

$$\min_{W\in\mathcal{M}_U} \|A - W\|_F.$$

In this paper, we propose the superoptimal preconditioners for some special functions of matrices. We first recall basic properties of the optimal and superoptimal preconditioners. Here, $\delta(A)$ stands for the diagonal matrix whose diagonal is equal to the diagonal of a matrix $A \in \mathbb{C}^{n\times n}$.

Lemma 1.1. ([10, 13–15]) *Let $A \in \mathbb{C}^{n\times n}$ and $\mathcal{W}(A) = \{\mathbf{x}^*A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n$ and $\|\mathbf{x}\|_2 = 1\}$, where $\|\cdot\|_2$ denotes the 2-norm. Then*

(i) *$c_U(A) = U^*\delta(UAU^*)U$ and is uniquely determined by A.*

(ii) *If both A and $c_U(A)$ are nonsingular, i.e., $0 \notin \mathcal{W}(A)$, then the superoptimal preconditioner $t_U(A)$ exists and is given by*

$$t_U(A) = c_U(AA^*)[c_U(A^*)]^{-1} = U^*\delta(UAA^*U^*)\delta^{-1}(UA^*U^*)U,$$

where A^* means the conjugate transpose of A.

(iii) *If both $c_U(A)$ and $t_U(A)$ are nonsingular, then one has*

$$\sigma_k\big([t_U(A)]^{-1}A\big) \le \sigma_k\big([c_U(A)]^{-1}A\big),$$

for $k = 1, \ldots, n$, where the singular values are in increasing order, $\sigma_1 \le \cdots \le \sigma_n$.

(iv) *If A is Hermitian positive definite, then one has*

$$\lambda_k\big([t_U(A)]^{-1}A\big) \le \lambda_k\big([c_U(A)]^{-1}A\big),$$

for $k = 1, \ldots, n$, where the eigenvalues are in increasing order, $\lambda_1 \le \cdots \le \lambda_n$.

(v) *Let $[M]_{kk}$ denote the k-th diagonal entry of a matrix $M \in \mathbb{C}^{n\times n}$. Then*

$$[UAA^*U^*]_{kk} \ge [UAU^*]_{kk} \cdot [UA^*U^*]_{kk} \ge 0,$$

for $k = 1, \ldots, n$.

In the following sections, we propose two superoptimal preconditioners for some special matrix functions f from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$: $t_U(f(A))$ and $f(t_U(A))$, where $A \in \mathbb{C}^{n\times n}$ is a given matrix. We discuss properties of both preconditioners $t_U(f(A))$ and $f(t_U(A))$ for different functions of matrices. We also report some numerical experiments for solving the system $f(A)\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \in \mathbb{C}^n$.