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Fully Discrete $\boldsymbol{A}$-ϕ Finite Element Method for Maxwell's Equations with a Nonlinear Boundary Condition

Tong Kang[1], Ran Wang[1], Tao Chen[1], and Huai Zhang[2,*]

[1] *Department of Applied Mathematics, School of Scineces, Communication University of China, Beijing, 100024, China.*
[2] *Key Laboratory of Computational Geodynamics, University of Chinese Academy of Sciences, Beijing, 100049, China.*

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Abstract. In this paper we present a fully discrete $\boldsymbol{A}$-ϕ finite element method to solve Maxwell's equations with a nonlinear degenerate boundary condition, which represents a generalization of the classical Silver-Müller condition for a non-perfect conductor. The relationship between the normal components of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ obeys a power-law nonlinearity of the type $\boldsymbol{H}\times\boldsymbol{n}=\boldsymbol{n}\times(|\boldsymbol{E}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{n})$ with $\alpha\in(0,1]$. We prove the existence and uniqueness of the solutions of the proposed $\boldsymbol{A}$-ϕ scheme and derive the error estimates. Finally, we present some numerical experiments to verify the theoretical result.

AMS subject classifications: 35Q60, 65N30, 65M60

Key words: Maxwell's equations; nonlinear Silver-Müller boundary condition; $\boldsymbol{A}$-ϕ method; full discretization; nodal elements; error estimates.

1. Introduction

We propose to study the following Maxwell's equations

$$\begin{cases}\partial_t(\varepsilon\boldsymbol{E})+\sigma\boldsymbol{E}=\nabla\times\boldsymbol{H}+\boldsymbol{J},\\ \partial_t(\mu\boldsymbol{H})+\nabla\times\boldsymbol{E}=\boldsymbol{0},\end{cases}\tag{1.1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{J}$ denotes the source current term. Let $\Omega\subset\mathbb{R}^3$ be a simply-connected bounded convex polyhedron with the boundary Γ, which consists of some conducting and nonconducting domains. Assume that there exist N faces $(F_j)_{j=1,\cdots,N}$ such that $\Gamma=\cup_j\overline{F}_j$. The electric permittivity ε, the electric conductivity σ and the magnetic permeability μ are supposed to be piecewise constants, and there exist constants ε_{min}, ε_{max}, σ_{max}, μ_{min} and μ_{max} such that $0<\varepsilon_{min}\le\varepsilon\le\varepsilon_{max}$, $0\le\sigma\le\sigma_{max}$ and $0<\mu_{min}\le\mu\le\mu_{max}$.

*Corresponding author. *Email address:* `huaizhang@gmail.com` (H. Zhang)

The problem (1.1) is accompanied with a nonlinear boundary condition between the normal components of $\boldsymbol{E}$ and $\boldsymbol{H}$, which corresponds to a non-perfect contact of different materials at the boundary. This means that the material on one side of the boundary doesn't allow the field to penetrate without losing the energy, which can be described in terms of an absorbing boundary condition. In this paper, we shall consider a power-law nonlinearity of the type

$$\boldsymbol{H}\times\boldsymbol{n}=\boldsymbol{n}\times\boldsymbol{g}(\boldsymbol{E}\times\boldsymbol{n})=\boldsymbol{n}\times(|\boldsymbol{E}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{n}),\qquad \alpha\in(0,1], \tag{1.2}$$

where $\boldsymbol{n}$ stands for the outward normal vector on the boundary. Then we obtain the following initial boundary value problem:

$$\begin{cases}\varepsilon\partial_{tt}\boldsymbol{E}+\sigma\partial_t\boldsymbol{E}+\nabla\times\left(\dfrac{1}{\mu}\nabla\times\boldsymbol{E}\right)=\boldsymbol{F}, & (\boldsymbol{x},t)\in\Omega\times(0,T],\\ \boldsymbol{n}\times\left(\dfrac{1}{\mu}\nabla\times\boldsymbol{E}\right)=\boldsymbol{n}\times\partial_t(|\boldsymbol{E}\times\boldsymbol{n}|^{\alpha-1}\boldsymbol{E}\times\boldsymbol{n}), & (\boldsymbol{x},t)\in\Gamma\times(0,T],\\ \boldsymbol{E}(\boldsymbol{x},0)=\boldsymbol{E}_0,\ \partial_t\boldsymbol{E}(\boldsymbol{x},0)=\boldsymbol{E}_0', & \boldsymbol{x}\in\Omega.\end{cases} \tag{1.3}$$

For a more general function $\boldsymbol{g}$, we have to adopt some assumptions ensuring the monotonicity, demicontinuity and coercivity of the nonlinear operator in appropriate function spaces. In the case when $\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{x}$, (1.2) represents the classical Silver-Müller condition, which (cf. e.g. [8,14]) is a first order approximation of the so-called "transparent" boundary condition. Sometimes it is also called Leontovich or impedance boundary condition (cf. e.g. [13,18]).

It is well-known that edge finite element methods and nodal finite element methods are usually used to approximate Maxwell's equations. Edge elements address the problem of discontinuity of the normal component of the field at the interface between two materials and avoid the nonphysical solutions called "spurious modes". There are a great deal of works on numerical approximation to Maxwell's equations and also on the convergence analysis and error estimates. For Maxwell's equations with perfectly conducting conditions, we refer readers to [6,10–12,19]. Numerically solving the full Maxwell's system with a linear Silver-Müller condition can been found in [9]. In [21,24,25], there have been some studies on quasi-static Maxwell's equations with a power-law nonlinear boundary. In [22], a mixed finite element scheme of the electric and magnetic field has been suggested for Maxwell's equations with a power-law nonlinear boundary condition.

To apply the nodal finite element method to solve Maxwell's equations, some potential fields are introduced and some gauges need imposing to guarantee the uniqueness of the potentials. The $\boldsymbol{A}$-ϕ finite element method (cf. e.g. [1–4,15–17,20]) is to decompose the electric field into summation of a vector potential and the gradient of a scalar potential, afterward to approximate the potential fields by piecewise polynomial functions. The $\boldsymbol{A}$-ϕ method has some advantages: First, although introducing the vector and scalar potentials increases the number of unknowns and equations, this seeming complication is justified by a better way of dealing with possible discontinuities