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Convergence and Quasi-Optimality of an Adaptive Multi-Penalty Discontinuous Galerkin Method

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Abstract. An adaptive multi-penalty discontinuous Galerkin method (AMPDG) for the diffusion problem is considered. Convergence and quasi-optimality of the AMPDG are proved. Compared with the analyses for the adaptive finite element method or the adaptive interior penalty discontinuous Galerkin method, extra works are done to overcome the difficulties caused by the additional penalty terms.

AMS subject classifications: 65N30, 65N12

Key words: Multi-penalty discontinuous Galerkin method, adaptive algorithm, convergence, quasi-optimality.

1. Introduction

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded, polyhedral domain. We study the convergence of an *adaptive multi-penalty discontinuous Galerkin* (AMPDG) method for the diffusion problem

$$L(u) := -\mathrm{div}(A\nabla u) = f \quad \text{in } \Omega, \tag{1.1a}$$
$$lu = 0 \quad \text{on } \partial\Omega. \tag{1.1b}$$

Precise conditions on $A : \Omega \to \mathbb{R}^{d\times d}$ and $f : \Omega \to \mathbb{R}$ are specified later. We would like to point out that, in the present work we restrict ourselves to homogeneous Dirichlet boundary conditions to simplify the already technical presentation. Similarly, a reaction term of the type cu with $0 \le c \in L^\infty(\Omega)$ could have been added to the development as in [10] without changing the essence.

The adaptivity has been a fundamental technique for about four decades in finite element methods (FEM), interior penalty discontinuous Galerkin (IPDG) methods, and many other methods to deal with various singularities. There have been many works

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on the convergence analysis of the adaptive FEMs (AFEM). It started with Babuška and Vogelius [2], who gave a detailed treatment for the one-dimensional boundary value problems. In 1996 Dörfler [14] introduced a crucial marking, from now on called Dörfler's marking, and proved the first convergent result for the two-dimensional case. He proved strict energy error reduction for the Poisson's equation provided the initial mesh satisfies a fineness assumption. For the results after that, we refer the reader to [10,13,17,18,20–23, etc.], and the references therein. Here we note that in [21,22], the MARK procedure of the adaptive algorithm marks not only for the estimator, but also for the oscillation, and the interior node property should be involved. However, in [10, 23] the impractical ingredients, that is, the mark of oscillation and the interior node property, were removed. This is a great improvement in a practical point of view.

Although there have been many works for the AFEM, the convergence results of the adaptive IPDG methods (AIPDG) are rather recent. The first convergence result on the AIPDG was given by Karakashian and Pascal [19], then Hoppe, Kanschat and Warburton [15] improved the results upon [19]. In 2010, Bonito and Nochetto [3] proved the convergence of the AIPDG for the diffusion problem with general data on nonconforming partitions. They also gave a quasi-optimal asymptotic rate of convergence for the AIPDG, which was the first result of this type in the literature for DG methods.

The multi-penalty discontinuous Galerkin (MPDG) method considered in this paper, which was first introduced by Arnold [1], penalizes not only the jump of discrete solution, but also the jump of the (higher) derivatives of the discrete solution at mesh interfaces. We point out that the latter one has successfully applied to convection-dominated problems as a stabilization technique [5–9], and more recently, it has shown great potential for simulating Helmholtz scattering problems with high wave number [25, 26].

The purpose of this paper is to prove convergence and quasi-optimality for the AMPDG based on an a posteriori error estimator of residual type and the Dörfler's marking strategy. The basic idea of the analysis is to mimic that of the AIPDG (cf. [3]), but some essential difficulties caused by the additional penalty terms need to be treated specially. Compared with [3], we introduce the extra penalty terms without any other restrictions for the data in model problem (1.1a) or the regularity of weak solution. Note that when the penalty parameters of the extra penalty terms equal to zero, the AMPDG reduces to the AIPDG, our results extend those of AIPDG [3].

The rest of this paper is organized as follows. In Section 2 we introduce the MPDG method and its adaptive algorithm, and give the preliminaries used to derive the main results of this paper. In Section 3 we give the upper and lower error estimates for the MPDG. Section 4 is devote to prove the contraction property of the adaptive algorithm. The quasi-optimality of the AMPDG is proved in Section 5.

In order to simplify the notation, we write $a \lesssim b$ whenever $a \leq Cb$ with a constant C independent of parameters which a and b may depend on. We also write $a \eqsim b$ for $a \lesssim b$ and $b \lesssim a$.