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# Comparison of Some Preconditioners for the Incompressible Navier-Stokes Equations

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**Abstract.** In this paper we explore the performance of the SIMPLER, augmented Lagrangian, 'grad-div' preconditioners and their new variants for the two-by-two block systems arising in the incompressible Navier-Stokes equations. The lid-driven cavity and flow over a finite flat plate are chosen as the benchmark problems. For each problem the Reynolds number varies from a low to the limiting number for a laminar flow.

**AMS subject classifications**: 65F10, 65F08

**Key words**: incompressible Navier-Stokes equations, two-by-two block systems, preconditioners.

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## 1. Introduction

In this paper we deal with efficient solution of the stationary, laminar incompressible Navier-Stokes equations, discretized by the Finite Element (FE) method. Due to the presence of the convective term, the Navier-Stokes problem is nonlinear and a suitable linearization technique is needed, like the Picard or Newton method [14]. Both linearizations result in solving a sequence of linear systems with the two-by-two block structure. Finding the solution of the linear system is the most time-consuming part of the numerical simulations. Taking into account the high consumption of the computational time and the memory storage by using the direct solution method, the Krylov subspace methods [1, 14, 31] become feasible to solve the large scale linear systems. It is widely recognized that preconditioning is the most critical ingredient in the development of efficient and reliable Krylov subspace methods.

In the past decades a number of preconditioners are proposed for the two-by-two block systems arising in the incompressible Navier-Stokes equations. In this paper we put our eyes on the block preconditioners, that are constructed by approximating the

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block factorization of the coefficient matrix and making full use of the available information about the geometry and physics of the problem. Many state-of-the-art block preconditioners have been devised, for example the SIMPLE-type preconditioners [20,35], the Pressure-Convection-Diffusion commutator (PCD) [18], the Least Squares commutator (LSC) [12], the augmented Lagrangian preconditioner (AL) [6] and the Gradient-Divergence ('grad-div') preconditioner [9,17]. For an overview of block preconditioners, we refer to [5,8,32,34].

Nowadays the AL and 'grad-div' preconditioners gain a lot of attention. In order to overcome the bottleneck of the AL preconditioner and make it more efficient, a modified version is devised [7]. The analysis between the AL and 'grad-div' preconditioners in this paper illustrate that the strategy leading to the modified AL preconditioner is still applicable in the 'grad-div' preconditioner. In this way, a modified variant of the 'grad-div' preconditioner is realised in this paper. The SIMPLE-type preconditioner remains attractive due to its simplicity and efficiency, and are widely utilised by engineers to solve the industry applications [19]. The improvements of the SIMPLE-type preconditioner are considered in this paper and the focus is on the numerical reliability and efficiency. A comparison between the three preconditioners with their variants is carried out on some academic benchmark problems in this paper. Numerical experiments show that all the improvements advanced in this work are successful, and the modified 'grad-div' preconditioner is the most efficient in terms of the computational time and memory storage.

The organization of the paper is as follows. In Section 2 we briefly state the problem formulation and the Newton and Picard linearization methods. The SIMPLER, augmented Lagrangian, 'grad-div' preconditioners and their variants are introduced in Section 3. Section 4 contains numerical illustrations and some conclusions are given in Section 5.

## 2. Problem formulation and linearization

A mathematical model for the incompressible flows reads as follows:

$$\begin{aligned} &-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f} && \text{on } \Omega,\\ &\nabla\cdot\mathbf{u}=0 && \text{on } \Omega,\\ &\mathbf{u}=\mathbf{g} && \text{on } \partial\Omega_D,\\ &\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}-\mathbf{n}p=0 && \text{on } \partial\Omega_N. \end{aligned} \tag{2.1}$$

Here $\mathbf{u}$ is the velocity, $p$ is the pressure and the positive coefficient $\nu$ is the kinematic viscosity, assumed here to be constant. Here $\Omega$ is a bounded and connected domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$, and $\partial\Omega=\partial\Omega_D\cup\partial\Omega_N$ is its boundary, where $\partial\Omega_D$ and $\partial\Omega_N$ denote the parts of the boundary where Dirichlet and Neumann boundary conditions for $\mathbf{u}$ are imposed, respectively. The terms $\mathbf{f}:\Omega\to\mathbb{R}^d$ and $\mathbf{g}$ are a given force field and Dirichlet