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A Non-Krylov Subspace Method for Solving Large and Sparse Linear System of Equations

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Abstract. Most current prevalent iterative methods can be classified into the so-called extended Krylov subspace methods, a class of iterative methods which do not fall into this category are also proposed in this paper. Comparing with traditional Krylov subspace methods which always depend on the matrix-vector multiplication with a fixed matrix, the newly introduced methods (the so-called (progressively) accumulated projection methods, or AP (PAP) for short) use a projection matrix which varies in every iteration to form a subspace from which an approximate solution is sought. More importantly an accelerative approach (called APAP) is introduced to improve the convergence of PAP method. Numerical experiments demonstrate some surprisingly improved convergence behavior. Comparison between benchmark extended Krylov subspace methods (Block Jacobi and GMRES) are made and one can also see remarkable advantage of APAP in some examples. APAP is also used to solve systems with extremely ill-conditioned coefficient matrix (the Hilbert matrix) and numerical experiments shows that it can bring very satisfactory results even when the size of system is up to a few thousands.

AMS subject classifications: 15A06; 65F10

Key words: Iterative method, accumulated projection, Krylov subspace.

1. Introduction

Linear systems of the form

$$Ax = b \tag{1.1}$$

where $A \in R^{n\times n}$ being nonsingular arise from tremendous mathematical applications and are the fundamental objects of almost every computational process. From the very ancient Gaussian elimination to the state-of-the-art methods like CG, MINRES, GMRES, as well as Multigrid method [1–3,14–16], numerous solvers of linear systems have been

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introduced and studied in extreme detail. Basically all solvers fall into two categories: direct methods and iterative methods.

Except for those specially designed methods for systems with some special properties, like symmetry, sparsity or triangularity, elimination methods based on LU factorization seem to be most widely accepted for general linear systems with satisfactory stability due to its flexibility of pivoting strategies [8, 9, 13]. Comparing with direct methods, iterative methods are a much larger family and have been accepting dominant attention. Since they make it possible for people to get a very 'close' solution to a system in much less arithmetic operation and storage requirement than direct methods and thus often lead to huge savings of time and costs.

Although some state-of-the-art direct methods can be applied to solve systems with pretty large amount of unknowns [3,10] in some situations, for even larger scale sparse systems (say, with unknowns up to a few millions) one can resort to the LGO-based solver [18, 19] recently introduced by authors, iterative methods are the only option available for many practical problems. For example, detailed three-dimensional multiphysics simulations lead to linear systems comprising hundreds of millions or even billions of equations in as many unknowns, systems with several millions of unknowns are now routinely encountered in many applications, making the use of iterative methods virtually mandatory.

The history of iterative methods can largely be divided into two major periods. The first period begins with 1850's while Jacobi and Gauss etc. established the first iterative methods named after these outstanding researchers and the period ends in 1970's. The majority of these iterative method are classified as stationary methods, which usually take the form:

$$x_{k+1} = Gx_k + v, \quad (k = 0, 1, 2, \cdots). \tag{1.2}$$

where v is a fixed vector and x_0 as the first guess. Excellent books covering the detailed analysis of error and convergence of these methods include works by Axelsson [2], Datta [7], Varga [22] and David Young [25], etc. The second period begins in the mid-1970s and is dominated by Krylov subspace methods and preconditioning techniques. Generally Krylov subspace methods use the following form

$$x_k = x_0 + y_k, \quad (k = 1, 2, \cdots) \tag{1.3}$$

where x^0 is an initial guess and y^k belongs to a so-called Krylov subspace

$$\mathcal{K}_m(G, v_0) \equiv span\{v_0, Gv_0, G^2v_0, \cdots, G^{m-1}v_0\}.$$

By assuming different strategies for seeking y_k from $\mathcal{K}_m(G, v_0)$, one gets a variety of iterative methods such as CG, BiCG, GMRES, FOM, MINRES, SYMMLQ, QMR [11, 17, 20, 21, 23], etc.

As a matter of fact, if we would refer extended Krylov subspace methods as those at each step of iteration the correction vector or approximate solution always comes from Krylov subspaces with a few fixed "generator" matrices (by a "generator" matrix