Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2016.y13024

Vol. **9**, No. 3, pp. 337-357
August 2016

Numerical Solution to the Multi-Term Time Fractional Diffusion Equation in a Finite Domain

Gongsheng Li*, Chunlong Sun, Xianzheng Jia, Dianhu Du

School of Sciences, Shandong University of Technology, Zibo, Shandong 255049, China.

Received 16 September 2013; Accepted 21 May 2015

Abstract. This paper deals with numerical solution to the multi-term time fractional diffusion equation in a finite domain. An implicit finite difference scheme is established based on Caputo's definition to the fractional derivatives, and the upper and lower bounds to the spectral radius of the coefficient matrix of the difference scheme are estimated, with which the unconditional stability and convergence are proved. The numerical results demonstrate the effectiveness of the theoretical analysis, and the method and technique can also be applied to other kinds of time/space fractional diffusion equations.

AMS subject classifications: 35R11; 65M06; 65M12

Key words: Multi-term time fractional diffusion; finite difference scheme; spectral radius; stability and convergence; numerical simulation.

1. Introduction

The partial differential equations of fractional order have played an important role in modeling of the anomalous phenomena and in the theory of the complex systems during the last two decades, see, e.g., [1–4,8,9,20,24,25]. The so-called time-fractional diffusion equation that is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ has to be especially mentioned. On the other hand, by the attempts to describe some real processes with the equations of the fractional order, several researches confronted with the situation that the order α of the time-fractional derivative from the corresponding model equations did not remain constant and changed, say, in the interval from 0 to 1, from 1 to 2 or even from 0 to 2. To manage these phenomena, several approaches were suggested. One of them introduces the fractional derivatives of the variable order, i.e., the derivatives with the order that can change with the time or/and depending on the spatial coordinates [5, 6, 16, 22], and the other way is to employ the multi-term time

*Corresponding author. *Email addresses:* ligs@sdut.edu.cn (Gongsheng Li), sunchunlong527@163.com (Chunlong Sun), kathy1978@126.com (Xianzheng Jia), dudianhu@126.com (Dianhu Du)

fractional differential equations. For $l > 0, T > 0$, the 1D multi-term time fractional diffusion equation in a finite homogeneous domain is given as

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \sum_{s=1}^{S} r_s \frac{\partial^{\beta_s} u}{\partial t^{\beta_s}} = D\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \ \ 0 < t < T, \tag{1.1}$$

where $u = u(x,t)$ denotes the state variable at space point x and time t, and α denotes the principal fractional order, and $\beta_1, \beta_2, \cdots, \beta_S$ are the multi-term fractional orders of the time derivatives, which satisfy the condition:

$$0 < \beta_S < \beta_{S-1} < \cdots < \beta_1 < \alpha < 1, \tag{1.2}$$

and $r_1, \cdots, r_S$ are positive constants, and $D > 0$ is the diffusion coefficient, $f(x,t)$ is a linear source term. All of the above time fractional derivatives are defined in the sense of Caputo, for example, the fractional derivative of the order $\beta \in (0,1)$ is given as

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta}. \tag{1.3}$$

See, e.g., Podlubny [23] and Kilbas *et al.* [11] for the definition and properties of the Caputo's derivative.

There are still a few research works reported on the multi-term time fractional diffusion equations like Eq. (1.1). On theoretical analysis and analytical methods, we refer to Daftardar-Gejji *et al.* [7], Luchko [19] and Jiang *et al.* [10]. In [7], the multi-term time-fractional diffusion-wave equation with the constant coefficients was considered, and a solution of the corresponding IBV problem was represented in form of the Fourier series via the multivariate Mittag-Leffler function. In [19], a generalized multi-term time fractional diffusion equation with the variable coefficients was considered, and well-posedness of the corresponding IBV problem was proved with the help of maximum principle together with the construction of solution's representation using the Fourier method. In [10], analytical solutions of the 1D multi-term time fractional diffusion-wave/diffusion equations were obtained also using the Fourier's separating variables method. However, numerical solutions for the multi-term time fractional diffusion equations are of the same importance as the analytical solutions, especially when deal with concrete computations for real diffusion phenomena. Recently in [15], Liu *et al.* considered numerical solution to the multi-term time fractional wave-diffusion equation, and put forward a fractional predictor-corrector method by transforming the equation into a system of time-fractional differential equations, and they gave the convergence rate of the proposed algorithm without discussing the stability.

In this paper, we continue to deal with numerical methods for solving Eq. (1.1) with the initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le l; \tag{1.4}$$

and the homogeneous boundary condition

$$u(0,t) = u(l,t) = 0, \quad 0 < t \le T, \tag{1.5}$$