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Convergence of an Anisotropic Perfectly Matched Layer Method for Helmholtz Scattering Problems

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Abstract. The anisotropic perfectly matched layer (APML) defines a continuous vector field outside a rectangle domain and performs the complex coordinate stretching along the vector field. Inspired by [Z. Chen *et al.*, Inverse Probl. Imag., 7, (2013):663–678] and based on the idea of the shortest distance, we propose a new approach to construct the vector field which still allows us to prove the exponential decay of the stretched Green function without the constraint on the thickness of the PML layer. Moreover, by using the reflection argument, we prove the stability of the PML problem in the PML layer and the convergence of the PML method. Numerical experiments are also included.

AMS subject classifications: 15A12; 65F10; 65F15

Key words: Anisotropic PML, Helmholtz equation, Reflection argument, Exponential convergence.

1. Introduction

We propose and study an anisotropic perfectly matched layer (APML) method for solving Helmholtz scattering problem with Dirichlet boundary condition

$$\Delta u + k^2 u = f \qquad \text{in } \Omega = \mathbb{R}^2 \setminus \bar{D}, \tag{1.1}$$

$$u = g \qquad \text{on } \Gamma_D, \tag{1.2}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \qquad \text{as } r = |x| \to \infty\,. \tag{1.3}$$

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Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $f \in (H^1(\Omega))'$ has the support inside $B(R_0) = \{x \in \mathbb{R}^2 : |x| \le R_0\}$, where $(H^1(\Omega))'$ being the dual space of $H^1(\Omega)$, and $g \in H^{1/2}(\Gamma_D)$. Here we assume the wave number k is a constant. We remark that the results in this paper can be extended to the case when $k^2(x)$ is a variable inside some bounded domains or to solve the scattering problems with other boundary conditions such as Neumann or impedance boundary conditions on Γ_D.

Since the pioneering work of the B$\acute{e}$r$\acute{e}$nger [1] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literatures (see, e.g, Turkel and Yefet [24], Teixeira and Chew [22] for their reviews). Of particular importance to the development and the analysis PML method is the technique of complex coordinate stretching by Chew and Weedon [12]. Under the assumption that the solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a fictitious layer of the finite thickness with specially designed model medium that absorbs all the waves that propagate from inside the computational domain.

The convergence of the PML method has drawn considerable attention in the literature. Lassa and Somersalo [17, 18] and Hohage *et al.* [15] studied the acoustic scattering problems for circular and smooth PML layers. The anisotropic PML method in which the PML layer is placed outside a rectangle or cuboid domain is of considerable interest as opposed to the circular PML method because it provides greater flexibility and efficiency to solve problems involving anisotropic scatters. The convergence of the uniaxial PML method has been considered recently in Chen and Wu [8], Kim and Pascial [16], Bramble and Pasciak [2] and Chen and Zheng [10] for the acoustic scattering problems. It is also proved in [2, 6, 8, 15–18] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In the practical application of the PML method, the adaptive PML method has been proposed and studied in Chen and Wu [7] for grating, Chen and Liu [6], and Chen and Wu [8] for acoustic scattering problem, Chen and Chen [3], and Chen *et al.* [4] for Maxwell problem. The main idea of the adaptive PML method is to use a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness the absorbing PML layers.

The purpose of this paper is to propose a new construction of the anisotropic PML method and prove the convergence of this APML method. We will extend the idea in [4] for electromagnetic scattering problems and in [5] for acoustic scattering problem. The main idea in [4,5] is to define a continuous vector field outside the computational domain and perform the complex coordinate stretching along the direction of the vector field. This construction of the PML method by performing the complex coordinate stretching along a continuous vector field is different from the uniaxial PML method