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Mesh Spacing Estimates and Efficiency Considerations for Moving Mesh Systems

Joan Remski*

Department of Mathematics and Statistics, The University of Michigan-Dearborn, Dearborn, MI 48128, U.S.A.

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Abstract. Adaptive numerical methods for solving partial differential equations (PDEs) that control the movement of grid points are called moving mesh methods. In this paper, these methods are examined in the case where a separate PDE, that depends on a monitor function, controls the behavior of the mesh. This results in a system of PDEs: one controlling the mesh and another solving the physical problem that is of interest. For a class of monitor functions resembling the arc length monitor, a trade off between computational efficiency in solving the moving mesh system and the accuracy level of the solution to the physical PDE is demonstrated. This accuracy is measured in the density of mesh points in the desired portion of the domain where the function has steep gradient. The balance of computational efficiency versus accuracy is illustrated numerically with both the arc length monitor and a monitor that minimizes certain interpolation errors. Physical solutions with steep gradients in small portions of their domain are considered for both the analysis and the computations.

AMS subject classifications: 65M50, 65M06
Key words: Numerical solution, partial differential equation, moving mesh method, monitor function.

1. Introduction

Moving mesh methods are a class of numerical techniques for solving partial differential equations (PDEs) where the mesh is allowed to evolve in order to better approximate the solution. The idea is to adaptively move grid points to places in the domain where the solution changes in a significant way, leaving fewer mesh points in regions where the solution does not change very much. The location of the grid points is determined by a mapping from a computational domain, Ω_C, into the physical domain where the PDE is originally defined, Ω_P ([1, 2]). There are several methods for determining the mesh mapping, but in this paper we focus on using a monitor function that

*Corresponding author. *Email address:* `remski@umich.edu` (J. Remski)

defines some measure of error, with the mesh mapping function distributing this error in some optimal way.

Using a monitor function is convenient, since the mesh transform can be defined as the solution of a PDE and solved simultaneously, along with the original PDE. This results in a moving mesh system of equations, with one PDE controlling the movement of the mesh points and another PDE defining the physical process of interest. While this coupled system incurs more computational cost, since it requires that the mesh points be tracked accurately, it avoids the use of interpolation on dynamically changing grids in the physical domain. Updating the mesh via such interpolation schemes may cause numerical instabilities and inaccuracies, especially when the mesh has large deformations.

Even though much work has been done in studying the individual components of the moving mesh system, few papers consider the computational properties of the entire system. In [3], the distribution of mesh points is viewed as a constraint on the physical PDE and the stability of the resulting system of DAEs is discussed. For physical problems with steep gradients or large function values, [4] considers the efficiency of the overall system and shows that for a moving mesh system using the arc length (AL) monitor, there is a gain in efficiency when compared with solving the original physical PDE on a fixed mesh. Since more mesh points are moved toward the region where the solution changes most, this moving mesh system gives a more accurate solution to the physical PDE, especially when compared to a fixed mesh solution. This shows that by the proper design of a moving mesh system, one can gain both accuracy and efficiency. The current paper extends the work done in [4] by establishing a similar efficiency result for a general class of monitor functions. Also, by considering a discretized version of the moving mesh system, we establish estimates that describe the accuracy of the computed solution in terms of the mesh spacing in the physical domain. In certain circumstances, these results allow the user to control the level of accuracy in the physical solution or the level of efficiency in the system by selecting an appropriate monitor function.

The choice of the monitor function plays a critical role in the grid distribution and is not always a simple decision [5]. The characteristics of the solution to the physical PDE, like large gradients or even singularities, usually determine which monitor functions are suitable. However, numerical complexities in the physical solution can result in similar complexities in the mesh equations, because the mesh information appears in the coefficients of the transformed physical PDE. For example, a physical solution with a large gradient may force the mesh to move more points to this region. But this uneven distribution can result in large gradients in the equation for the mesh transform. So while the moving mesh solution gains accuracy in regions where the physical solution exhibits great change, the coupled system of physical PDE plus mesh PDE may have added computational cost.

In this paper, we examine a set of monitor functions that resemble the AL monitor ([1, 2]) and a monitor that optimizes interpolation errors (OI) [1]. While these monitor functions share a similar form and seem to affect the qualitative behavior of the grid