Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2016.m1603

Vol. **9**, No. 4, pp. 497-527
November 2016

Adaptive Mixed GMsFEM for Flows in Heterogeneous Media

Ho Yuen Chan[1,*], Eric Chung[1] and Yalchin Efendiev [2]

[1] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong SAR*
[2] *Department of Mathematics and Institute for Scientific Computation (ISC), Texas A&M University, College Station, Texas 77843-3368, USA.*

Received 8 January 2016; Accepted (in revised version) 6 July 2016

Abstract. In this paper, we present two adaptive methods for the basis enrichment of the mixed Generalized Multiscale Finite Element Method (GMsFEM) for solving the flow problem in heterogeneous media. We develop an a-posteriori error indicator which depends on the norm of a local residual operator. Based on this indicator, we construct an offline adaptive method to increase the number of basis functions locally in coarse regions with large local residuals. We also develop an online adaptive method which iteratively enriches the function space by adding new functions computed based on the residual of the previous solution and special minimum energy snapshots. We show theoretically and numerically the convergence of the two methods. The online method is, in general, better than the offline method as the online method is able to capture distant effects (at a cost of online computations), and both methods have faster convergence than a uniform enrichment. Analysis shows that the online method should start with a certain number of initial basis functions in order to have the best performance. The numerical results confirm this and show further that with correct selection of initial basis functions, the convergence of the online method can be independent of the contrast of the medium. We consider cases with both very high and very low conducting inclusions and channels in our numerical experiments.

AMS subject classifications: 65N30, 65N12

Key words: Mixed multiscale finite element methods, multiscale basis, adaptivity, online basis, flow in heterogeneous media.

1. Introduction

Many real-world problems involve multiple scales and high contrast. To solve these problems, we often adopt some forms of model reduction such as upscaling and

*Corresponding author. *Email addresses:* `hychan@math.cuhk.edu.hk` (H. Y. Chan), `tschung@math.cuhk.edu.hk` (E. Chung), `efendiev@math.tamu.edu` (Y. Efendiev)

multiscale methods. These methods can reduce the degrees of freedom of a problem. For example, in upscaling methods [17, 22, 25, 28], the multiscale media are upscaled so that the problem can be solved on a coarse grid. In multiscale methods [1, 3, 5, 6, 8, 9, 12, 16, 18–21, 23, 24, 27], basis functions are solved on a fine grid to capture the multiscale features of a medium and the problem is then solved on the coarse grid with these basis functions.

In this paper, we will present two adaptive enrichment algorithms for the generalized multiscale finite element method (GMsFEM) in solving the mixed framework of the flow problem in heterogeneous media [10]. The first method is based on a local error indicator. We use this indicator to search for the regions, where more basis functions are needed. This method will only add pre-computed basis functions, which are computed in the offline stage so we call it an offline adaptive method. In the second method, new basis functions are computed based on the previous solutions. We call it an online adaptive method.

GMsFEM [7] is a generalization of the classical multiscale finite element method [26]. In the classical method, one basis function per coarse edge is used to capture the multiscale features. For the multiscale mixed finite element method, one may see [1, 2, 4]. GMsFEM allows more basis functions per coarse edge to be used to take into account the effects of non-separable scales. The main idea is to solve local spectral problems for the selection of basis functions. The formation of basis functions in GMsFEM can be divided into offline and online stages. In the offline stage, offline basis functions are computed based on the multiscale features so that these functions can be reused for any input parameters to solve the equation. Online functions are those depending on the parameters. In [15], an adaptive algorithm is developed to enrich the space by adding basis functions which are formed in the offline stage. In [14], adaptive methods which involve the formation of new online basis functions based on the previous solution are developed. These methods show significant acceleration in the convergence rate of GMsFEM. There are also related methods developed for the discontinuous Galerkin formulation in [11] and [13].

In the paper, we will focus on the mixed framework of the flow problem. The mixed methods are important for many applications, such as flows in porous media, where the mass conservation is essential. We developed two adaptive methods to enrich the function space. One involves only offline basis functions while the other adds new online basis functions that are constructed using special minimum energy snapshots. We call them an offline and an online adaptive methods respectively. Two local spectral problems are developed for constructing multiscale basis functions. Both of them can be used in the online method, but only one can be used in the offline method. We propose error indicators which are based on the $\mathcal{L}^2$ and the $H(\text{div})$ norms of the local residual. These error indicators can be used to approximate the error of the solution. From [10], we know that the error between the GMsFEM solution and the fine grid solution involves two parts: one due to the selection of the basis functions and the other due to the discretization of the source function. In this paper, we will assume the error due to the discretization of the source function is small and consider only the