Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2016.m1405

Vol. **9**, No. 4, pp. 528-548
November 2016

Mixed Finite Element Methods for Fourth Order Elliptic Optimal Control Problems

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Received 22 January 2014; Accepted (in revised version) 27 April 2016

Abstract. In this paper, a priori error estimates are derived for the mixed finite element discretization of optimal control problems governed by fourth order elliptic partial differential equations. The state and co-state are discretized by Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. The error estimates derived for the state variable as well as those for the control variable seem to be new. We illustrate with a numerical example to confirm our theoretical results.

AMS subject classifications: 65N15, 65N30

Key words: fourth order mixed finite element methods, optimal control problems, L^2 projection, a priori error estimates.

1. Introduction

During the last decade the discretization of optimal control problems involving second-order elliptic partial differential equations (introduced in [Lions (1971); Tröltzsch (2010)]) has a number of applications in mathematical and physical problems. For instance, heat conduction, diffusion, electromagnetic waves, fluid flows, freezing processes, and many other physical phenomena can be modeled by partial differential equations. As far as numerical approximation of control problems is concerned, finite element methods (FEMs) play a vital role since these methods have certain advantages over finite difference methods. In many optimal control problems, the objective functional contains not only the primal state variable but also its gradient. The advantage of mixed element methods is that the approximations to u and the flux $\boldsymbol{p}$ can be obtained simultaneously. There are several results available in the literature in which FEMs are used for the numerical approximation of optimal control problems (see [10–12, 14]). Recently there appeared some new progresses on the numerical solution of optimal control problems; this include, the work of Hinze in [17] in which he introduced a new variational discretization approach for linear-quadratic optimal control problems and

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obtained improved convergence order for optimal control in [Hinze (2005)], where the control set is not discretized explicitly. The problems described by bi-harmonic equations arise from fluid mechanics and solid mechanics such as bending of elastic plates. From an application point of view, the interest in higher order elliptic equations as constraints in optimization problems is two fold. First, of course, is the use of plate models as constraints in optimization. Further applications occur in fluid mechanics in the context of the stream function formulation of the Navier-Stokes equations. Second, mixed formulations of fourth order problems have a close connection to necessary conditions for optimization problems governed by second order partial differential equations.

The choice of a mixed discretization of the biharmonic problem is thus, on one hand, motivated by the connection to the bi-level optimization case and on the other hand by the possibility to approximate this using H^1-conforming finite elements instead of the more expensive H^2-conforming elements. There has been much research about mixed finite element methods for the 4th order PDEs, for example, Ciarlet-Raviart elements, Herrmann-Miyoshi elements, Hellan-Herrmann-Johnson elements found in (see [4, 9, 19, 27, 29]). In [3], the author has presented estimation of the control error in discretization PDE-constrained optimization. Further the author has estimated the error in the control variable measured in a natural norm. In [23], the author has established vorticity superconvergence of a finite element method for the biharmonic equation by Ciarlet-Raviart's scheme under the biquadratic uniform rectangular mesh. In [21], the author has developed mixed finite element methods for the fourth-order nonlinear elliptic problem. Optimal L^2 error estimates are proved by using a special interpolation operator on the standard tensor-product mixed finite element methods of order $k \geq 1$. In [20], the author has studied a mixed finite element method for the optimal boundary control problem governed by the bi-harmonic equation. The system of optimality equations consisting of state and costate functions is derived. Further for optimality equation based on the system, a gradient-type optimization method is used as a mixed finite approximation for solving the optimal boundary control problem.

Recently in [8], the author has studied a priori error estimates of Ciarlet-Raviart mixed finite element methods for fourth order elliptic control problems with the first biharmonic equation. The optimality conditions consisting of the state and the costate equations are derived. In [18], the author has developed priori error estimates for the finite element discretization of optimal distributed control problems governed by the biharmonic operator. The state equation is discretized in primal mixed form using continuous piecewise biquadratic finite elements, while piecewise constant approximations are used for the control. Further error estimates are derived for the state variable as well as those for the control which are order-optimal on general unstructured meshes. However, on uniform meshes, not all error estimates are optimal due to the low-order control approximation. In [7], the author has developed posteriori error estimates for nonconforming finite element methods for fourth-order problems on rectangles. In [26], the author has presented sufficient conditions for the existence of at least one nontrivial weak solution to a fourth order elliptic problem with a $p(x)$-biharmonic operator and the Navier boundary conditions. In [2, 13, 15, 24, 25], the