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The Gradient Superconvergence of Bilinear Finite Volume Element for Elliptic Problems

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Abstract. We study the gradient superconvergence of bilinear finite volume element (FVE) solving the elliptic problems. First, a superclose weak estimate is established for the bilinear form of the FVE method. Then, we prove that the gradient approximation of the FVE solution has the superconvergence property:

$$\max_{P\in S}|(\nabla u-\overline{\nabla} u_h)(P)| = O(h^2)|\ln h|,$$

where $\overline{\nabla} u_h(P)$ denotes the average gradient on elements containing point P and S is the set of optimal stress points composed of the mesh points, the midpoints of edges and the centers of elements.

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1. Introduction

The finite volume element (FVE) method has been widely used in numerically solving partial deferential equations. The main feature of FVE method is that it inherits some physical conservation laws of original problems locally, which are very desirable in practical applications. During the last decades, many research works have been presented for FVE methods solving various partial differential equations, see [1–10, 12–15, 17, 18, 20] and the references cited therein.

Superconvergence of numerical solutions has been an active research area for finite element method (FEM) since its practical importance in enhancing the accuracy of finite element approximation. But the study of superconvergence properties of FVE methods is far behind that of FEMs. For elliptic problems in two-dimensional domain,

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the early superconvergence results of linear and bilinear FVE solutions are of this form [3,10]

$$\Big(\frac{1}{N}\sum_{z\in S}|(\nabla u-\overline{\nabla}u_h)(z)|^2\Big)^{\frac{1}{2}}\le Ch^2\|u\|_{3,\infty}, \tag{1.1}$$

where S is the set of optimal stress points of interpolation function on partition T_h, $N=O(h^{-2})$ is the total number of points in S, and $\overline{\nabla}$ denotes the average gradient on elements containing point z. Later, Lv and Li in [12] extended result (1.1) to the isoparametric bilinear FVE on quadrilateral meshes under the h^2-uniform mesh condition. Recently, Zhang and Zou in [20] also derived some superconvergence results for the bi-complete k-order FVE on rectangular meshes, and in the case of bilinear FVE $(k=1)$, their result is

$$|\nabla(u-u_h)(G)|\le Ch^2|\ln h|^{\frac{1}{2}}\|u\|_{4,\infty}, \tag{1.2}$$

where G is the Gauss point of element (the midpoint of element). Moreover, by using the postprocessing technique, Chou and Ye [7] obtain the superconvergence estimate:

$$\|\nabla u-\nabla Qu_h)\|\le Ch^{\frac{3}{2}}\|u\|_3, \tag{1.3}$$

where Qu_h is the postprocessed linear FVE solution obtained by the L_2-projection method; Zhang and Sheng [18] further derive the superconvergence estimate:

$$\|\nabla u-R\nabla u_h)\|\le Ch^2\|u\|_3, \tag{1.4}$$

where $R\nabla u_h$ is the reconstructed gradient of bilinear FVE solution obtained by using the patch interpolation recovery method.

In this paper, we consider the bilinear FVE method to solve the following problem

$$\begin{cases} -div(A\nabla u)+c\,u=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.5}$$

where $\Omega\subset R^2$ is a rectangular domain with boundary $\partial\Omega$, coefficient matrix $A=(a_{ij})_{2\times 2}$. Our main goal is to give some piecewise-point gradient superconvergence for the bilinear FVE approximation to problem (1.5). To the authors' best knowledge, in existing literatures, only the midpoints of elements are proved to be the superconvergence point of gradient approximation [20] (also see (1.2)), we here will prove that except the midpoints of elements, all interior mesh points and midpoints of interior edges are also the superconvergence points. Generally speaking, the analysis of bilinear FVE on rectangular meshes is more difficult then that of linear FVE on triangle meshes, the reason is that ∇u_h is not constant in the former case. By calculating exactly some integrals on element and its boundary, we first establish the superclose weak estimate for the bilinear form of the FVE method,

$$\begin{aligned}&|a_h(u-\Pi_h u,\Pi_h^* v)|\le Ch^2\|u\|_{3,p}\|v\|_{1,q},\\ &\qquad\forall\, v\in U_h,\ 2\le p\le\infty,\ \frac{1}{p}+\frac{1}{q}=1,\end{aligned} \tag{1.6}$$