Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2016.m1503

Vol. **9**, No. 4, pp. 595-618
November 2016

Second Order Convergence of the Interpolation based on Q_1^c-Element

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Received 6 March 2015; Accepted (in revised version) 18 February 2016

Abstract. In this paper, the second order convergence of the interpolation based on Q_1^c-element is derived in the case of d=1, 2 and 3. Using the integral average on each element, the new basis functions of tensor product type is builded up and we can easily extend it to the higher dimensional case. Finally, some numerical tests are made to show the analytical results of the interpolation errors.

AMS subject classifications: 35Q30, 65N30

Key words: Q_1^c-finite element, Tensor-product polynomial space, Integral average, Non-uniform mesh, Interpolation error.

1. Introduction

The interpolation theory of functions is very important in the numerical analysis of finite element methods [1]. In this paper, we establish the local interpolation error estimates of Q_1^c-element for the scalar-valued functions living in the Sobolev spaces.

Given a set of d intervals $\{[0, L_i]\}_{1\leq i\leq d}$, the set $K = \prod_{i=1}^d [0, L_i]$ is called a cuboid. Let Q_1 be the polynomial space in the variables $x_1, \ldots, x_d$, with real coefficients and of degree at most 1 in each variable. For $x \in K$, there exists a unique vector $(t_1, \ldots, t_d) \in [0, 1]^d$ such that, for all $1 \leq i \leq d$, $x_i = t_i L_i$. In the case of dimension 1, $Q_1 = P_1$; in the case of dimension $d \geq 2$,

$$Q_1 = \left\{ p(x) : p(x) = \sum_{0\leq i_1,\ldots,i_d\leq 1} \alpha_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d} \right\}.$$

One readily verifies that Q_1 is a vector space with 2^d dimensions. Obviously, Q_1 is the tensor product Lagrange finite elements. Let $\{T^h\}$, $0 < h \leq 1$, be a non-degenerate

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rectangular family of subdivisions of K. Setting $\widetilde{I}_h$ be the interpolation operator constructed by Q_1. Its sharper interpolation error estimate is also well-known [1–3]. If the function u is smooth enough, i.e., $u \in H^2(K)$, then

$$\|u-\widetilde{I}_h u\|_{0,K} + h|u-\widetilde{I}_h u|_{1,K} \le ch^2|u|_{2,K}.$$

This estimate is optimal. Here and below $c>0$ is a constant independent of h. $\|\cdot\|_{i,K}$ and $|\cdot|_{i,K}$ denote the norm and semi-norm respectively on the standard Sobolev space $H^i(K)$ with $i=0,\ 1,\ 2$.

On the other hand, we know that the approximation of order k can be achieved as long as polynomials of degree k are used [1,4,5]. These results apply to both the standard tensor-product polynomial spaces and the serendipity elements as well. However, one may wonder what the fewer extra terms in the tensor-product polynomial spaces can be added to obtain the approximate accuracy of $k+1$ order. In this paper, we consider a new polynomial space Q_1^c by using Q_1 conforming element and the fewer extra basis functions to enhance the approximation order in case of R^d. Here d quadratic polynomials of each direction variable are added in R^d space. In other words, for the case of dimension 1, the function space is

$$Q_1 \oplus \text{span}\Big\{-\frac{h_i^2}{2}\phi_{i-1}\phi_i\Big\}.$$

For the case of dimension 2, the basis function space is

$$Q_1 \oplus \text{span}\Big\{-\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j\Big\}.$$

For the case of dimension 3, the basis function space is

$$Q_1 \oplus \text{span}\Big\{-\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j,\ -\frac{\kappa_k^2}{2}\theta_{k-1}\theta_k\Big\}.$$

Here $h_i,\ \tau_j$ and κ_k are step sizes of each element in directions x, y and z respectively. The basis functions $\phi_i(x)$, $\psi_j(y)$ and $\theta_k(z)$ are linear. Setting I_h be the interpolation operator constructed by Q_1^c. In the cases of dimension 1 and 2, if $u\in H^3(K)$, then the following interpolation error estimate holds

$$\|u-I_h u\|_{0,K} + h|u-I_h u|_{1,K} + h^2|u-I_h u|_{2,K} \le ch^3|u|_{3,K},$$

(see Theorem 2.1 and Theorem 3.1 for details). In dimension 3, if $u\in H^4(K)$, then the following interpolation error estimate also holds

$$\begin{aligned}&\|u-I_h u\|_{0,K} + h|u-I_h u|_{1,K} + h^2|u-I_h u|_{2,K}\\ \le & c\Big(h^3|u|_{3,K} + h^4|u|_{4,K}\Big),\end{aligned}$$

(see Theorem 4.1 for details).