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A three-dimensional model of $SL(2,\mathbb{R})$ and the hyperbolic pattern of $SL(2,\mathbb{Z})$

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Abstract

The special linear group $SL(2,\mathbb{R})$, the group of 2×2 real matrices with determinant one, is one of the most important and fundamental mathematical objects not only in mathematics but also in physics. In this paper, we propose a three-dimensional model of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$, which is realized by embedding $SL(2,\mathbb{R})$ into the unit 3-sphere. In this model, the set of symmetric matrices of $SL(2,\mathbb{Z})$ forms a hyperbolic pattern on the unit disk, like the islands floating on the sea named $SL(2,\mathbb{R})$. The structure of this hyperbolic pattern is described in the upper half-plane H. The upper half-plane H also enables us to generate symmetric matrices of $SL(2,\mathbb{R})$ with three circles. Furthermore, the well-known fact $H = SL(2,\mathbb{R})/SO(2)$ is visualized as S^1 fibers of Hopf fibration in the unit 3-sphere. With this three-dimensional model in $\mathbb{R}^3$, we can have a concrete image of $SL(2,\mathbb{R})$ and its noncommutative group structure. This kind of visualization might bring great benefits for the readers who have background not only in mathematics, but also in all areas of science.

1 Introduction

The purpose of this paper is to propose a three-dimensional model of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$. The special linear group $SL(2,\mathbb{R})$, the group of 2×2 real matrices with determinant one, is one of the most important and fundamental mathematical objects not only in mathematics (see, [7, 9, 10]) but also in physics (see, [1, 5]). Nevertheless, it is difficult for us to grasp the whole image of $SL(2,\mathbb{R})$ and its noncommutative group structure. The three-dimensional model of $SL(2,\mathbb{R})$ is realized by embedding $SL(2,\mathbb{R})$ into the unit 3-sphere. By the stereographic projection from the unit 3-sphere into $\mathbb{R}^3$, we can visualize every element in $SL(2,\mathbb{R})$ as a point in $\mathbb{R}^3$. In this three-dimensional model, the set of symmetric matrices of $SL(2,\mathbb{Z})$ forms a hyperbolic pattern on the unit disk as shown in Figure 1. This hyperbolic pattern is regarded as a visualization of the well-known fact $H = SL(2,\mathbb{R})/SO(2)$, where H is the hyperbolic plane and $SO(2)$ is the special orthogonal group in dimension 2.

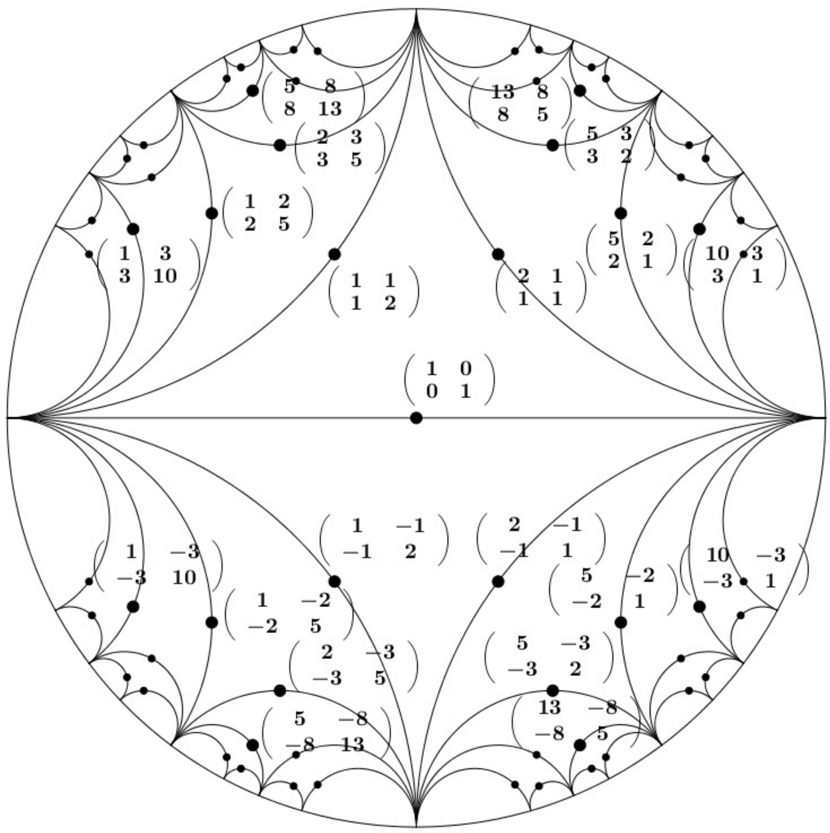

Figure 1: Hyperbolic pattern of $SL(2, \mathbb{Z})$.

In Section 2, we construct the three-dimensional model of $SL(2, \mathbb{R})$ in $\mathbb{R}^3$. In Section 3, we focus on the hyperbolic pattern of the set of symmetric matrices of $SL(2, \mathbb{Z})$. Finally, the well-known fact $H = SL(2, \mathbb{R})/SO(2)$ is visualized as S^1 fibers of Hopf fibration (see, [3] pp.320–323, [4] pp. 298–305) in the model in Section 4.

2 Three-dimensional model of $SL(2, \mathbb{R})$

In this section, we propose a three-dimensional model of $SL(2, \mathbb{R})$. The real special linear group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}) \,\middle|\, ad - bc = 1 \right\}$$

is embedded into the three-dimensional unit sphere

$$S^3 = \left\{ (u, v) \in \mathbb{C}^2 \,\middle|\, |u|^2 + |v|^2 = 1 \right\}.$$

To see this, let C_0 be a great circle in S^3 defined by

$$C_0 = \left\{ (0, e^{i\theta}) \in S^3 \,\middle|\, \theta \in [0, 2\pi) \right\}.$$