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A Reflection on Teaching Mathematical Proofs to First-Year Undergraduates

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Abstract

Writing a mathematical proof is an essential skill for qualified science, technology, engineering, and mathematics (STEM) students, especially those majoring in mathematics, data science, artificial intelligence (AI), and computer science. However, many first-year students complain that it is tremendously difficult for them to learn mathematics well, particularly mathematics proofs. New teaching staff are also often frustrated during their first couple of years of teaching undergraduates mathematics, wondering why something so seemingly simple is so difficult to teach. Although most universities have experienced this problem, there has been little deep reflection and investigation on why learning of mathematical proofs is so difficult for first-year undergraduates and how to mitigate such difficulties. The aim of this paper is to fill such a gap by reflections and investigation which focus on the recognition and learning of the mathematical proof process.

Keywords: formal proof, transition, beliefs, academic performance, undergraduate mathematics

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Importance of Mathematical Proofs

Role of Proofs

A proof is a valid argument that establishes the truth of a (mathematical) statement. An argument is a compound statement with some premises (assumptions or facts) and a conclusion. According to this definition, a proof is a way of generating new knowledge based on existing knowledge or assumptions. Almost all new mathematical results are established by a rigorous mathematical proof. A mathematical proof is not only important to establish significant results as theorems, but also for many other applications. It can be used to design a debugging system for computer programs, to make inferences in artificial intelligence (AI), to show whether a system specification is consistent, and so on. It is well accepted that a mathematical proof is one of the foundations of computer science and AI.

Proving is seen as an essential skill for science, technology, engineering, and mathematics (STEM) students to acquire. Bell (1976) argues that a mathematical proof can be used for verification, illumination, and systematization. It can also be used for discovery or communication or simply as an intellectual challenge (De Villiers, 1990, 2003). Balacheff (2010) points out that a mathematical proof can also be used for the construction of empirical theory, exploration of the meaning of a definition or the consequences of an assumption, and the incorporation of a well-known fact into a new framework, thus enabling it to be viewed from a fresh perspective.

Teaching Proofs Is Important

Teaching proofs is not only important for students majoring in STEM, but for almost any subject, as has been discussed by many researchers. We believe that teaching mathematics properly is critically important for the following reasons.

First, writing a proof helps students to understand basic definitions, concepts, and theory. A proof can help students learn about why a theory holds from some basic or simple facts. Once students understand the reason and mechanism underlying a proof, they can remember results more deeply and for longer, as the learning of mathematics usually requires students to both understand and memorize. As memorization with proper understanding can accelerate students' learning (Zhang et al., 2019), a proof can help students to deepen their understanding of underlying mathematical ideas and reinforce their memorization (Lesseig et al., 2019).

Second, a proof can help students to build up a knowledge graph (KG) which is easily taken away. A KG is like a bunch of grapes (Abu-Salih, 2021). A graph node in the KG can be seen as an important concept, definition, or theorem, while a proof works as a hidden edge between grapes. Proofs can group several important "grapes" together and then form a large bunch of KGs. It is much easier to take away a bunch of grapes than a few scattered ones. The same holds with remembering knowledge: It is much easier to remember a lot of material systematically if one can build up a KG rather than remember each knowledge point separately.

Third, proofs can improve students' problem-solving skills and analytic thinking. As Rav (1999) suggested, a mathematical proof embodies tools, methods, and strategies for solving problems which enable students to achieve self-learning. We usually need proofs to obtain important results. That is, we usually need to break up a complicated problem into several parts, such as premises and a conclusion, premises being the resources available and the conclusion being our target. A proof is our strategy, effort, and solution to achieve the goal with given resources. If we cannot achieve our targeted results directly with available resources (premises), we have to bridge these premises with our targeted results. Such a bridge is a built by using constructive indirect strategies, which often require knowledge other than the underlying premises. To obtain our targeted results, we have to use the resources (knowledge) we have. Whether or not can one prove the result largely depends on the knowledge one has at one's command and what kind of resources one can provide. Such a process, that breaks a complicated problem down into small pieces and then solves it constructively, whether directly or indirectly, is, in fact, the essence of analytic thinking, creative analysis, and problem solving. Therefore, learning to write a proof develops students' mathematical reasoning (Hanna, 1995) as well as their analytical skills.

Fourth, teaching proofs can train students in the language of mathematics and logic. Mathematics is seen as a language (or discourse) (Sfard, 2008). However, unlike the daily language that everyone speaks, the language of mathematics is a selective skill which can only be obtained via formal training, unlike in music or the arts, where people can appreciate a work without knowing the underlying language (for example, notes). It is hard for people to appreciate a piece of math work without knowing proofs. They usually obtain this mathematical language skill by formal training, and students learn the language of mathematics largely through proofs. Teaching proofs actually provides the most natural way of cultivating the mathematical language in higher-level mathematics. The constitution of mathematical proofs cannot be apart from the signs and symbols used in proofs. The teaching and learning of proofs engages students in authentic mathematical practices, revealing the axiomatic structure of the discipline and the infallible nature of mathematical truths (Zaslavsky et al., 2012); this is the power and rigor of mathematical language.

Teaching proofs can help students to create new knowledge. Once students have learned sufficient knowledge and become fluent in the language of mathematics, they can use the analytic-thinking and problem-solving skills on which they have been trained to create new knowledge. There is no doubt that almost all new mathematics is established and created via proofs. Hemmi (2006) adds that "transfer" can be another function of proof in mathematics teaching and learning, as firstly, "working with proofs can be useful in another contexts than in mathematics," and secondly, "some proofs can provide methods or techniques useful in other mathematical contexts" (p. 223).

Teaching and Learning Mathematical Proofs Is Challenging

Having shown, in the above, that teaching and learning proofs is of great importance for STEM education, the significance of overcoming the first-year difficulties in learning proofs is obvious. To ensure the sustainable and effective learning of mathematics proofs, it is essential to reflect on the difficulties in practice of teaching them to first-year undergraduates and their difficulties in learning it. Difficulties in regard to teaching mathematical proofs to junior undergraduates can arise from the following aspects: students, teachers and curriculum design.

Difficulties for Students

The first year of university life is a critical transition period for most students. The most challenging transition for students might be that they need to quickly adapt their learning style from passive to active. For most Asian students, university life usually means that they have to begin to live and learn independently. At high schools, they have intensive pressure from parents, guardians, and teachers; thus, many people are around them to motivate them to learn. When students enter universities, in contrast, there will be less pressure from parents or guardians. Some students even feel that they are escaping the overbearing influence of their parents. Staff in universities prefer to provide support when students turn to them; otherwise, they would feel as if they are interfering in a student's life. It takes most students time to learn self-management and self-motivated learning properly.

Another fact is that the learning content grows exponentially between high school and university. Students have to manage a couple of math courses and several other courses, which might also be a challenge for them. For example, in mathematical analysis I (or the simpler version of calculus I), linear algebra is scheduled for the first semester in most Chinese universities. In the second semester, besides mathematical analysis II and linear algebra II, which are both fundamental courses, a third mathematics course – analytical geometry – is often scheduled simultaneously. Such first-year college mathematics courses cover far more content than everything learned in the previous dozen years of K12 education. It is easy for most students to lose balance while managing so many courses simultaneously in such a transition period and facing this pace of information explosion. If students are weak in self-management, in particular, they can easily be frustrated by certain courses.

From the Perspective of Teachers

Novice teachers often experience difficulties delivering mathematical proofs. They, too, are in a transition period, from researchers or PhD students to lecturers. Correspondingly, their audience changes from domain-specific experts, reviewers, or collaborators to new undergraduates. They do not usually have enough time to get to know their new audience well. Some basic content knowledge which lecturers think trivial for domain-specific experts does not, in fact, come naturally to first-year students.

They have to explain in detail from the perspective of their new audience rather than their old one. Most teachers are told that teaching should be delivered in a student-centered way. However, they often base their teaching on their own high school or university experiences from several years ago. Their memory of being at university might have faded, if they can even remember what they have learned in high school. Students' background has, in fact, usually changed considerably due to K12 education reform.

Another challenge that teachers have to face is diversity in students' background. This is especially important for Chinese universities, due to possible imbalances between education in different regions. Students come from different provinces and have various educational backgrounds. Although the college entrance examination tries to homogenize such backgrounds, educational disparities between the east and west and between the cities and the countryside exist. Such disparities are more significant in some universities, such as the authors' university, which pays much attention to education equality in future. How to manage such a diversity of backgrounds is a challenge when personal resources are limited, so it is a challenge to teach mathematical proofs.

A third challenge for teachers is the limited time available to deliver a very full syllabus. With the rapid development of technology and advancement of knowledge, more courses are proposed to be taught in universities, which has definitely reduced the teaching hours available for classical courses such as mathematics. For example, mathematical analysis used to be taught over four semesters (usually of 20 weeks) and two years in the 1980s, but now it is compressed into three shortened 16-week semesters. However, the basics for subsequent courses must still be covered, which means it is difficult to shorten the content of the syllabus. Under such time pressure, most teachers would give up rigor proof on certain math results. Hence, students may use some math theorems without properly understanding them. Math students are treated as engineering students, and this touches on the deepest issue regarding how to select teaching content and curriculum.

Curriculum Design and Syllabus

Although we have emphasized the importance of mathematical proofs, many mathematical curricula omit the basics of proportional logic and inference rules, which are critically important to construct a proof. For example, most first-year courses start with calculus, mathematical analysis, or linear algebra, none of which covers proportional logic. One obstacle for students learning calculus or mathematical analysis is the ϵ–δ language, which is an advanced topic in propositional logic.

It takes a couple of weeks to cover this advanced topic, from the basics through proposition, logic connectives, compound propositions, truth table, implications, propositional variables, and qualifiers for compound propositions. A proposition is a statement which is true or false and is the basis of propositional logic. If a proposition is like a note or a key in music, then the ϵ–δ language is like an Italian opera. We cannot expect our first-year students to sing a opera without knowing all the keys and notes! It takes many years for the most brilliant mathematician to master the ϵ–δ language, while we expect our students to pick it up without any training! For so many years, there few teachers to pay such an attention.

Learning Process Analysis: A Case Study

The above explanations give our readers a sense of how difficult it can be when students enter universities for them to learn mathematical proofs. In this part, we report on two specific students who learned mathematical proofs in the first year. Through interacting with the students about their learning experiences, we aimed to gain a more comprehensive understanding of the difficulties experienced by first-year undergraduates in learning mathematical proofs.

Background

A brief introduction to the two students' background is given first, and technical terms are clarified. The two students selected as cases in this study both finished their upper secondary school in China and passed the university entrance exam. The students are from the same university, and both are majoring in mathematics and applied mathematics (MAM). Their university is a public university in China which is in both Project 211 and Project 985. These programs tend to develop and reinforce some specific high-level universities and subjects to which particular importance is attributed. MAM is one of main majors in the Department of Mathematics, which offers courses on "pure mathematics" with the aim of nurturing students with a strong mathematical ability and background. "Pure mathematics" here refers to Hardy's (1992) description of the "function of a mathematician [being] to do something, to prove new theorems, to add to mathematics" (p. 61). The core courses for the students in the first year are mathematical analysis, advanced algebra, and geometry.

Data Collection and Analysis

This case study is based on the data collected through observation and interviews with the two first-year undergraduate students. The whole study began with a short survey which aimed to learn more about the students' background and experience of learning, and how they view mathematical proofs. Some of the questions in the survey are adapted from Almeida (2000) and Hemmi (2006).

Following the short survey, a semi-structured interview was held with the two students ("SA" and "SC"). The interview is in three parts to understand their experiences of learning proofs. The first part includes several open-ended questions to examine how they think about proofs; the second part considers the transition process to investigate how students experience the change in learning proofs from secondary school to university; and the third part includes some practical exercises related to mathematical proofs and formal definitions. The main proof used as an example in this study is taken from the mathematical analysis course, which is taught in two semesters as "Analysis 1" and "Analysis 2".

Detailed interpretive analysis of these materials helped us to identify the specific issues that students may encounter and recognize possible causes of these difficulties, which may provide significant implications for the teaching and learning of mathematical proofs. The aim described above also leads the study to a qualitative research methodology, as the main goal is to document the details of data and interpret them appropriately. The qualitative analysis enabled us to understand what or how students experience in the process of learning, both physically and mentally; their interpretations also reveal how they describe their feelings (Lapan et al., 2012). Below is the proof example used in this study and the related mathematical knowledge that would be used in solving this problem.

Proof Example

Let f : R→R be a continuous function. If $\lim_{x\to+\infty} f(x)=+\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$, then prove that there exists at least one point c such that $f(c) = 0$ (Boero, 2015).

Note: main knowledge used in the study is listed below:

The Definition of Limit (The Limit of a Function)

Let f be a function defined on some open interval that contains x_0, except possibly x_0 itself. Then we say the limit of $f(x)$ as x approaches to x_0 is A and write as

$$\lim_{x\to x_0}[f(x)]=A,$$

if for every $\varepsilon > 0$, there exists , $|x - x_0| < \delta$, such that

$$|f(x) - A| < \varepsilon.$$

The Definition of Continuous Function

A function $f(x)$ is continuous at the point $x = x_0$, if $\lim_{x\to x_0}[f(x)]$ exists and is equal to $f(x_0)$. A function $f(x)$ is continuous if it is continuous at all points in its domain.

Intermediate Value Theorem (IVT)

Let $f(x)$ be a continuous function on $[a,b]$ and suppose $f(a) < y < f(b)$, then there exists an $x = c$ with $c \in [a,b]$ such that $f(c) = y$.

Introduction to Cases

Student Case 1 ("SA")

When SA is doing this example, she first thinks of using IVT. In order to find the closed interval required in IVT, she says, "I want to get the value half by half, because the function $f(x)$ is continuous, there must exist, such that y equals to 1, in a similar way, you can find a y equals to −1, then you have got the closed interval" (A1).

SA applies the features of continuous

function by specifying two constant points in the function. However, she does not mention anything about the two given limits. Therefore, when the researcher (the first author) asks her if she is convinced by this proof, she says, "well, if I were required to do it by myself, this is usually the way I find the interval. I am not sure whether this is the right way, but I am used to doing it this way" (A2).

Then she looks back at her proof, thinks for a while, and says that "it does not seem to work very well, so can you remind me of the definition of continuous function?" (A3).

In the above proving process, the student then chooses to check the definition of continuous function. After we provide her with this definition, she tries to discuss it with the researcher:

> Can I just pick an x_0, and discuss the different cases with it, if $f(x_0)$ equals to 0, $x_0$would be c, but if it is not, then double x_0 to try again and repeat this process, until I find the right c, I think I must be able to find this point, because the function is continuous, is this right? (A4)

Then the researcher reminds her that she has been given two limits by the context of the problem. She instantly realizes she must use the definition of limit and finds the closed interval, saying:

> Well, because of the limit, given that $A > 0$, there must exist a such that for every $x > a$, $f(x) > A$, then take one $a1 > a$, $f(a_1) = A_1 = A$, similarly, $f(b_1) = B_1 < B$; therefore, I shall find the closed interval $[b_1, a_1]$, then I can use the IVT and get c in $[b_1, a_1]$, this works well. (A5) [1]

Finally, the student seems to believe that this is the way to prove this problem. She is also satisfied with this proof.

Student Case 2 ("SC")

The second student (SC) also faces difficulties during her work with the proof example:

> It looks like using the IVT, so I think I can make it in this way, as $\lim_{x\to+\infty} f(x) = +\infty$, there must exist $a > 0$, such that $f(a) = A > 0$, and in a same manner, there should also be a number b, such that $f(b) = B < 0$, therefore, the case is totally same to that in the IVT and the problem is proved. (C1)

The student thinks of the IVT immediately, which shows she has the sense that the formal IVT theorem is applied in this proof. Then the researcher asks her whether she is completely convinced by the

[1] The student writes down the formulas as she says.

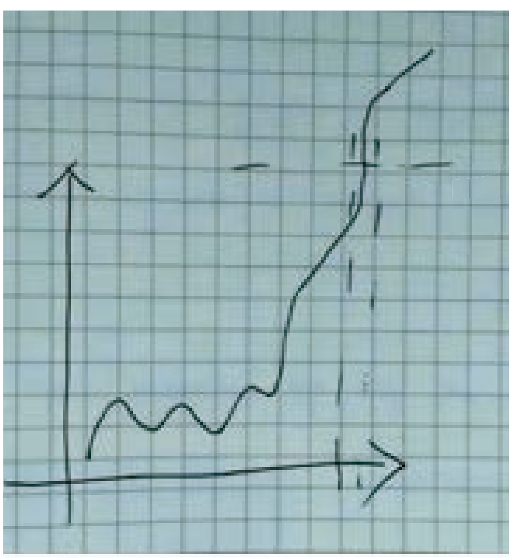
proof above. The student looks through her proof again and answers yes (C1).

The researcher asks her, "why do you think that if $\lim_{x \to +\infty} f(x) = +\infty$, there must exist $a > 0$ such that $f(a) = A > 0$?"

She thinks for a while and says, "I am sure about it; look, if the limit does exist, there must have an $a > 0$ and $f(a) = A > 0$." She draws a line (Figure 1) and shows it to the researcher (C2). After that, she refuses to do more with the example and insists, "that's all for this problem" (C3).

Figure 1

Student C's Draft of a Continuous Function



Although the researcher tries to remind her about the main concepts in this problem and the key points, she refuses to continue this task, insisting that there is nothing wrong with her proof, and a and b are there, which is obvious. It is also worth noting that at the beginning, SC gives a and b in a very direct way, without showing their development. In other words, like SA, at the beginning, SC depends on an intuitive and vague understanding of the whole problem to prove.

Case Reflections

Combining students' performances in the proof example and their responses in the interviews, possible reflections about teaching and learning proof may be raised, as discussed in the following sections.

Concept Understanding

To compare the two students, as shown in the cases, SA successfully managed the formal mathematical proof in the end, while SC failed to give an appropriate formal proof for this problem. In SC's problem-solving process, she shows a strong invalid belief about her own proof. That is to say, in her perception, a mathematical proof is exactly what she does. The problems revealed by this case are similar to what Hawro (2007) assumes: "Conjecture 3: The students did not know how, or did not feel the need, to use the definition to evaluate the veracity of subsequent steps in the analyzed proof" (C1 and C3); "Conjecture 4: Whilst doing the proof, the students did not refer to the definition of a concept but to the content of the concept image" (C2); and "Conjecture 5: The students did not know what constitutes the proof" (C2; pp. 2294-2295).

Moore (1994) illustrates in detail that the main difficulty for students to produce a proof is the concept understanding, which has three parts: concept definitions, concept images, and concept usage. In this concept understanding scheme, concept definition means the formal verbal definition that specifically and precisely defines the concept in "a noncircular manner", while the concept image is defined by associations of the properties and mental pictures which characterize the concept (Tall & Vinner, 1981). Another aspect is concept usage, which refers to how individuals apply and construct the concepts in generating examples or doing proofs (Moore, 1994).

Moore (1994) has illustrated that there are two main reasons for students' difficulties with definitions. One is the abstraction of these concepts, which, in this case, both SA and SC mention in their interviews. For example, SC said:

> I feel that "limit" has become complicated since entering university, you know, in secondary school, the notion of "limit" is very clear and simple, there is nothing confused, but in the university, in the first session when introducing the concept, it is taught in a way with integration and derivation which makes it more complex. Besides, it is expressed in a manner which is also very hard to understand. Especially when doing exercises related with limit, I always have the feeling that I need to know more or there is something missed. (C4)

The other reason is that students may not have a sense that there is a great distinction between formal definition and informal explanation based on concept image, and SC specifically reflects on this reason. Admittedly, SC's intuitive reference and the overall idea of her proof can be seen as correct, but she has not applied any formal language or rules of inference a formal proof would require. She seems to believe that an explanation is adequate if it makes sense, whether or not it follows logical rules. The importance of formal language and careful reasoning which may be embedded in the concept definition has been neglected. The concept image of SC does help the student to understand the problem and build the logic of the proof, but it is not acceptable as a formal proof in university mathematics learning with a high level of rigor (Moore, 1994).

Although informal approaches in mathematics are essential and helpful when encountering a proof, they are not able to guarantee that students can produce correct proofs. On the one hand, within the increased level of rigor of university mathematics, the students must acquire the ability to use formal definitions. On the other hand, a concept image can never be enough in the proving process. Firstly, it lacks the formal language which is used to express a proof; secondly, it cannot supply the individual steps in proofs; and thirdly, it cannot expose the logic in the structure of a proof, whereas definitions do (Moore, 1994). When it comes to concept usage, in the example of this case, the definitions of "limit" and "continuous function" not only play roles in justifying every step in the proof by supplying mathematical language and verbal expressions, but also help to build the whole logical structure of the proof. However, as Conjecture 5 says, SC does not realize that the definition of limit in the example actually connects the proof and has to be constituted in the whole proof.

Reliance and Uncertainty

From SA's reflections (A2), it is fair to suggest that various reasons drive her to consider her proof a second time. In the first place, she would like to involve the researcher, rather than write the proof independently. In this proving process, she believes that she can seek help and instruction from the researcher. Therefore, the student shows a reliance on teachers or tutors when doing mathematical proofs. Furthermore, she admits that she relies not only on teachers or instructions from other people, but also on textbooks, answer sheets, and any other related materials:

> Sometimes, if I am not able to figure it out, I would like to ask teachers for help or maybe refer to the answer sheets ... this is because that they may help to remind me of something useful ... I also refer to the answers given for some exercises. Sometimes, I find that the proof process from this step to the next step is difficult to continue in my mind, then I look back the content in the textbook and notice that there is a theorem embedded in this step but I forgot it or I did not notice it before. (A6)

Thus, the question is to what extent these instructions can really help students to develop their mathematics learning, especially in proving. On the one hand, with these instructions, students may be able to become used to the strategies and concepts applied in mathematical proofs. On the other hand, it is also possible that this reliance may not promote their learning of how to do mathematical proofs but, rather, hinder the process: If they rely on other help, it is likely that they may lack the self-developing process, then they are unable to find the necessary used theorems or definitions in the proof next time, when they prove it independently.

The second reason that drives the student to prove again is the uncertainty about her proof. According to Zaslavsky et al. (2012), uncertainty "motivate(s) people to change or expand their existing ways of thinking" (p. 223). Therefore, it drives the need for proof. It is reasonable to assume that if the student is certain about her proof, she may not change her mind and insist that her proof is right. In this case, the sources of the "uncertainty" come not only from the problem itself but also the proof that the student has just done. That is to say, the student is not sure about her own proof. In this sense, the reason the student feels dissatisfied with her own proof, and the reason she still proves it in her own way, even though she is not sure about it, become two points in the proving process.

Viewing the strategies that SA uses in both A1 and A4 to find the closed interval required in the IVT theorem, it is clear that the student tries two methods, both of which give the interval in an informal way, thus without using any formal definition and merely based on her own understanding. However, each time the researcher asks again whether she likes her proof, she does not answer. It is reasonable to suggest that she is still not very confident with her proving process in A1 and A4. Possible reasons for the lack of confidence may be that, first, as a first-year student, she lacks experience of doing mathematical proofs, so she may not be able to judge what is a good and acceptable proof here. Another is that she may be clear that a formal proof here is essential, but she has no idea how to prove it in a formal way, so comes up with an informal one.

Experience

In the process SA follows to write the proof, once she is given the hint to use the definition of limit, she instantly figures the problem out and shows strong satisfaction with the final proof (A5). That is to say, she accepts the need for a formal proof, and she thinks that only when the proof follows the logical routine and laws and uses the formal concepts and theorem in a reasonable way can it be seen as convincing. However, in the practical exercises, she may not always be able to live up to the rigor of the process. Furthermore, for some reason, when she is struggling with proofs, she may also try to use an informal proof which may be intuitively right.

Sierpinska (2000) demonstrates that mathematicians apply two kinds of thinking modes in doing proofs. The first is called theoretical thinking, which is identified by an organizational system which includes various concepts, as well as reflection on the semiotic means of representation. Another mode of thinking is called practical thinking, which, in contrast to the first, is an application of "prototypical examples, reasoning based on the logic of action" (Gueudet, 2008, p. 241).

However, because of a lack of both mathematical experience and comprehensive knowledge organization, first-year mathematics students are not able to develop a proof based on a theoretical understanding. SA, as a first-year mathematics student, shows the process whereby a novice student, when given instructions, gradually changes her mind about her proof and finally develops it through theoretical concepts and theorems.

Beliefs

In the semi-structured interview, when the researcher asks the two students about what mathematics is, SA answers that mathematics is "a tool, a hobby, a research and a direction." She explains that:

> When I consider mathematics as a tool, it is used to solve problems. Mathematics can also be seen as a hobby because some people may really enjoy doing mathematics, therefore, it is like an interest such as music or sport. Besides, some people may aim at being mathematicians who are knowledgeable in mathematics and devoted to mathematics; mathematics for those people would be a research area. (A7)

She addresses the fourth idea, "direction," by saying that mathematics is a direction which leads students to learn more about it so as to develop it as a career:

> Like, those who are interested in analyzing data may be interested in being an analyst. So I think, in the future, those learning mathematics will either be data analysts or mathematicians who are doing mathematical research in solving mathematics hypotheses, prove theorems etc. (A8)

When talking about proof learning at university (in the first year), SA says that, "usually it [proof] is used to prove that the function is convergent, or uniform conver gent, or others, I mean, to prove something is true. It might be a way to accumulate experiences to solve some hypothesis" (A9).

SC explains that:

> I think mathematics is a basic subject. … Learning mathematical proof is to

> learn how to do the exercises and get great grades in the exams. … It [proof] is a method to generalize the individual process which makes problem solving simple. ... It can be used to refine one process for more conditions, as in the Chinese saying "Ju Yi Fan San" ("when you understand one example, you will be able to understand other similar ones"). (C5–C7)

Judging whether the two students' opinion of mathematics and mathematical proof is comprehensive or not is one question; however, as concerns the relationship between them, SA views mathematical proof as also a method to drive the development of mathematics (A8 and A9), while SC views it as a basic subject. She explains that "compared with other subjects, such as physics and chemistry, mathematics is a 'basic' subject, and different to other disciplines." Mathematical proof is also seen as a strategy to do exercises and solve problems.

These ideas reflect two different beliefs about mathematics learning as well as mathematical proofs. First, formal proofs play an important role in mathematics research, being its final stage, so in this process, a formal proof refines the argument and gives it in a deductive way, according to mathematical deductions as well as precise definitions. Besides, Boas (1981) claims that "only mathematicians learn from proofs, other people learn from explanations, a great deal can be accomplished with argument that fall short of proofs" (p. 729). In this sense, SA captures the perception of mathematical proofs in a reasonable enough way. On the other hand, for SC, the learning of mathematics seems to be to meet teaching aims and pursue good grades. The significance of learning mathematical proofs or the crucial rules that mathematical proofs play in mathematics is not valued. In this case, SC shows a relatively narrow conception of the nature of mathematics as well as mathematical proofs.

Research has shown that students' positive attitude towards mathematics can positively influence their performance. Therefore, a narrow belief may not only be a hurdle for students to develop their mathematics competency but also hinders the improvement in their ability to do proofs, since a limited view of mathematical proofs may affect the extent to which the students really understand the role of proofs. It is very likely that a passive attitude is one of the obstacles for students to improve their mathematics skills, including doing mathematical proofs. Thus, research is necessary into how these beliefs appear and to what extent they influence the learning of mathematics among students.

Discussion and Conclusions

Through providing a case study of students' experiences of mathematical proof learning, this paper reflects on the teaching and learning of formal proofs by first-year undergraduates at university. The above discussion and analysis raise certain questions for us to rethink the teaching and learning of mathematical proofs.

Difficulties in Secondary–Tertiary Transition

The different performances of the two students considered in regard to doing mathematical proofs show students' difficulties in doing such proofs from various aspects. First, the perception of a formal proof may be one of the most significant factors which determine how far a student will be able to manage to do such a proof. In this study, SA has a more complete perception of a formal proof than SC, with an agreed sense that in university, a mathematical proof concerns writing a "formal" proof rather than just being descriptive. It is also reasonable to suggest that if a student does not have a comprehensive understanding of a formal proof, then, to a large extent, they may not able to do one, since a sense of the formal helps to determine whether the proof can be accepted.

Apart from the perception of formal proofs, a comprehensive understanding in terms of mathematical concepts is crucial. As illustrated above, involving concept definition, concept image, and concept usage in the concept understanding scheme can be seen as a core strategy, which is also a main part in doing formal proofs at university. Therefore, it turns out to be more important for students to acquire a helpful concept-understanding scheme. For instance, both the students considered in this study feel difficulty when encountering the formal concept of "limit." At the end of the proof, SA actually grasps the precise language and reasoning inference in the concept definition and combines them with her concept image, which helps her obtain the idea to get the closed interval in IVT. Then, through combining all these materials, the student finally shows the formal proof. However, because of a lack of concept understanding, which specifically means the incomplete understanding of formal concept definition, SC does not finally show the proof in a precise way.

Furthermore, both the students are first-year undergraduate mathematics students, and in interacting with mathematical proofs, one of the main obstacles recognized by several studies is lack of experience. Accumulating experience of doing formal proofs plays an important role in the development of a mathematician. It fosters the flexibility to combine different reasoning and thinking modes (Gueudet, 2008). Therefore, long-term interaction with formal proofs may help students formulate not only a mature view of formal proofs but also a deep concept understanding in different aspects.

As seen from the case of the two students, it is reasonable to suggest that possible affective factors may also influence students mathematics learning in university, particularly on the learning of proofs. For example, from our study, students' beliefs about mathematics, their emotions during learning, and their reliance towards instructors may all affect their learning experiences. However, to date, most researchers may focus on the area of elementary and secondary level mathematics learning, more research should be done to further investigate how learning of proofs are intertwined with learners' emotional experiences.

Implications for Teaching and Learning

The passage from secondary to tertiary mathematics has been recognized by a great variety of research (e.g., Gueudet, 2008). It is also agreed that students may be faced with various changes in the learning of mathematics in this period. Research conducted in a Western context has shown that students have various difficulties in doing mathematical proofs, for instance their beliefs about proofs, their difficulties in constructing proofs, their understanding of abstract logic, and their problems with formal mathematics language (de Guzmán et al., 1998; Gueudet, 2008; Moore, 1994; Selden, 2012). Turning to the Chinese education context, although there has been an increase in research of higher-level mathematics education, less research has specifically considered the transition process. There are various reasons for this; for instance, there has been more of a tradition of addressing students' academic performance in compulsory education than when they enter universities (Leung, 1992). Moreover, compared with school-level education, tertiary-level mathematics may be more complicated with regard to students' various needs and bases. Therefore, this study, considering students' experience of doing formal proofs in the first year in university and their discourses about the transition to proofs, which can be seen as a pivot study, provides a way to connect research on tertiary-level education and elementary education in China.

As illustrated earlier in this paper, from the perspectives of students, teachers, and curriculum design, various obstacles may influence students' learning of mathematical proofs. This study has the following implications to improve the effectiveness of mathematics teaching and learning at university and help students overcome the difficulties of doing mathematical proofs.

Firstly, mathematical proofs at tertiary level require great rigor, as they involve understanding and applying formal concepts and theorems as well as constructing the proper logic (e.g., Tall & Vinner, 1981). Students then need to train themselves in such processes. Students' processes of doing mathematical proofs can be vital resources for analyzing their learning conditions and difficulties, and more attention should be paid to students' learning experiences in this transition period. However, research rarely investigates students' difficulties in this learning period, and more research should be conducted to understand students' learning difficulties.

Second, seen from the teacher's perspective, university lecturers are usually pure mathematics researchers or PhD students who are working on transferring from the role of researcher to that of teacher. On the one hand, they may not be familiar with students' learning experiences as that learning stage was too long ago for them; on the other hand, a lack of teacher training and experiences may not facilitate effective teaching. Therefore, we encourage university lectures in mathematics to reflect on teaching of mathematics proofs. For example, it might be helpful to build the connections between students mathematics learning and real-world context, such as engineer, economics and so on, for students, through which students may see the effectiveness of mathematical proofs in other subjects that are more relevant to their everyday life.

From the perspective of curriculum design, as proofs are becoming more formal than what students encounter in upper secondary school, it should be noted that certain supports, from various aspects, should be offered to help students learn more fluently. For example, transitional courses in terms of formal language, mathematical logic, and so on may help students to master the transition process, but such courses are rarely offered in universities. Other than additional courses, it would be also interesting to look at how different pedagogic approaches would influence students learning of proofs in university. Future empirical studies should be encouraged on this topic.

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