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Rigidity of Minimizers in Nonlocal Phase Transitions II

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Abstract. In this paper we extend the results of [12] to the borderline case $s=\frac{1}{2}$. We obtain the classification of global bounded solutions with asymptotically flat level sets for semilinear nonlocal equations of the type

$$\Delta^{\frac{1}{2}} u = W'(u) \quad \text{in } \mathbb{R}^n,$$

where W is a double well potential.

Key Words: De Giorgi's conjecture, fractional Laplacian.

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1 Introduction

We continue the study initiated in [12] for the classification of global bounded solutions with asymptotically flat level sets for nonlocal semilinear equations of the type

$$\Delta^s u = W'(u) \quad \text{in } \mathbb{R}^n,$$

where W is a double well potential.

The case $s\in(\frac{1}{2},1)$ was treated in [12] while $s\in(0,\frac{1}{2})$ was considered by Dipierro, Serra and Valdinoci in [5]. In this paper we obtain the classification of global minimizers with asymptotically flat level sets in the remaining borderline case $s=\frac{1}{2}$. All these works were motivated by the study of semilinear equations for the case of the classical Laplacian $s=1$, and their connection with the theory of minimal surfaces, see [2,4,9,10]. It turns out that when $s\in[\frac{1}{2},1)$, the rescaled level sets of u still converge to a minimal surface while for $s\in(0,\frac{1}{2})$ they converge to an s-nonlocal minimal surface, see [13].

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We consider the Ginzburg-Landau energy functional with nonlocal interactions corresponding to $\Delta^{1/2}$,

$$J(u,\Omega)=\frac{1}{4}\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\complement\Omega\times\complement\Omega)}\frac{(u(x)-u(y))^2}{|x-y|^{n+1}}dxdy+\int_\Omega W(u)dx,$$

with $|u|\le 1$, and W a double-well potential with minima at 1 and -1 satisfying

$$\begin{aligned}&W\in C^2([-1,1]),\quad W(-1)=W(1)=0,\quad W>0\quad \text{on}\quad (-1,1),\\&W'(-1)=W'(1)=0,\quad W''(-1)>0,\quad W''(1)>0.\end{aligned}$$

Critical functions for the energy J satisfy the Euler-Lagrange equation

$$\Delta^{1/2}u=W'(u),$$

where $\Delta^{1/2}u$ is defined as

$$\Delta^{1/2}u(x)=PV\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|y-x|^{n+1}}dy.$$

Our main result provides the classification of minimizers with asymptotically flat level sets.

Theorem 1.1. Let u be a global minimizer of J in $\mathbb{R}^n$. If the 0 level set $\{u=0\}$ is asymptotically flat at ∞, then u is one-dimensional.

The hypothesis that $\{u=0\}$ is asymptotically flat means that there exist sequences of positive numbers θ_k, l_k and unit vectors ξ_k with $l_k\to\infty$, $\theta_k l_k^{-1}\to 0$ such that

$$\{u=0\}\cap B_{l_k}\subset\{|x\cdot\xi_k|<\theta_k\}.$$

By saying that u is one-dimensional we understand that u depends only on one direction ξ, i.e., $u=g(x\cdot\xi)$.

As in [12], we obtain several corollaries. We state two of them.

Theorem 1.2. A global minimizer of J is one-dimensional in dimension $n\le 7$.

Theorem 1.3. Let $u\in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta^{1/2}u=W'(u), \tag{1.1}$$

such that

$$|u|\le 1,\quad \partial_n u>0,\quad \lim_{x_n\to\pm\infty}u(x',x_n)=\pm 1. \tag{1.2}$$

Then u is one-dimensional if $n\le 8$.

Theorem 1.2 and Theorem 1.3 without the limit assumption in (1.2), have been established by Cabre and Cinti [1] in dimension $n=3$. Recently, Figalli and Serra [6] obtained the same conclusion for all stable solutions in dimension $n=3$. Their result, combined with Theorem 1.2 above, implies the validity of Theorem 1.3 without the limit assumption in (1.2), in dimension $n=4$.

We prove our result by making use of the extension property of $\Delta^{1/2}$. Let $U(x,y)$ be the harmonic extension of $u(x)$ in $\mathbb{R}^{n+1}_+$

$$\Delta U=0 \quad \text{in } \mathbb{R}^{n+1}_+, \quad U(x,0)=u(x),$$

then

$$\Delta^{1/2}u(x)=c_n U_y(x,0),$$

with c_n a dimensional constant. Global minimizers of $J(u)$ in $\mathbb{R}^n$ with $|u|\le 1$ correspond to global minimizers of the extension energy $\mathcal{J}(U)$ with $|U|\le 1$, where

$$\mathcal{J}(U):=\frac{c_n}{2}\int|\nabla U|^2 dxdy+\int W(u)dx.$$

After dividing by a constant and relabeling W we may fix c_n to be 1.

We obtain Theorem 1.1 from an improvement of flatness property for the level sets of minimizers of $\mathcal{J}$, see Proposition 6.1. We follow the main steps from [11,12], however some technical modifications are required. The main difference when $s=\frac{1}{2}$ is that at a point of $\{u=0\}$ which has a large ball of radius R tangent from one side we can no longer estimate its curvatures in terms of R^{-1}. Instead we obtain an integral estimate (see Lemma 3.3) which turns out to be sufficient for the key Harnack estimate of the level sets.

The paper is organized as follows. In Sections 2 we introduce some notation and then construct a family G_R of axial supersolutions. In Sections 3 and 4 we provide viscosity properties for the mean curvature of the level set $\{u=0\}$. In Section 5 we obtain the Harnack inequality of the level sets and in Section 6 we prove our main result by compactness.

2 Supersolution profiles

We introduce the following notation.

We denote points in $\mathbb{R}^n$ as $x=(x',x_n)$ with $x'\in\mathbb{R}^{n-1}$. The ball of center z and radius r is denoted by $B_r(z)$, and $B_r:=B_r(0)$. Points in the extension variables $\mathbb{R}^{n+1}_+$ are denoted by $X=(x,y)$ with $y>0$, and the ball of radius r as $\mathcal{B}_r^+$

$$\mathcal{B}_r^+:=\{(x,y)\in\mathbb{R}^{n+1}_+ \,|\, |(x,y)|<r\}\subset\mathbb{R}^{n+1}.$$

Given a function $U(x,y)$ we denote by $u(x)$ its trace on $\{y=0\}$.

Let $\mathcal{J}$ be the energy

$$\mathcal{J}(U,\mathcal{B}_R^+):=\frac{1}{2}\int_{\mathcal{B}_R^+}|\nabla U|^2dxdy+\int_{B_r}W(u)dx,$$

and a critical function U for $\mathcal{J}$ satisfies the Euler-Lagrange equation

$$\Delta U=0,\quad U_y=W'(u). \tag{2.1}$$

In [8] it was established the existence and uniqueness up to translations of a global minimizer G of $\mathcal{J}$ in 2D which is increasing in the first variable and which has limits ± 1 at $\pm\infty$:

a) $G:\mathbb{R}^2_+\to(-1,1)$ solves the Eq. (2.1),

b) $G(t,y)$ is increasing in the t variable and its trace $g(t):=G(t,0)$ satisfies

$$g(0)=0,\quad \lim_{t\to\pm\infty}g(t)=\pm 1.$$

Moreover, g and g' have the following asymptotic behavior

$$1-|g|\sim\min\{1,|t|^{-1}\},\quad g'\sim\min\{1,|t|^{-2}\}, \tag{2.2}$$

and

$$\mathcal{J}(G,B_R^+)=C^*\log R+\mathcal{O}(1),$$

for some constant C^*.

Constants that depend on n, W, G are called universal constants, and we denote them by C, c. In the course of the proofs the values of C, c may change from line to line when there is no possibility of confusion. If the constants depend on other parameters, say θ, ρ, then we denote them by $C(\theta,\rho)$ etc.

For simplicity of notation we assume that

$$W \text{ is uniformly convex outside the interval } [g(-1),g(1)]. \tag{2.3}$$

Since $\Delta G_t=0$ and $G_t\geq 0$, we easily conclude that

$$|G_y|\leq C\frac{1}{1+r},\quad C'\frac{1+y}{1+r^2}\geq G_t\geq c'\frac{1+y}{1+r^2}, \tag{2.4}$$

where r denotes the distance to the origin in the (t,y)-plane. We also obtain

$$|G|\leq 1-cr^{-1},\quad \forall r\geq 1.$$

Define in $\mathcal{B}^+_{R/8}$,

$$H_R(t,y):=G(t,y)+\frac{C_0}{R}\left((y+C_1)\log R-y\log(y+1)+\frac{1}{R}(t^2-y^2)\right), \tag{2.5}$$

for some C_0, C_1 universal, large to be made precise later.

Let $\bar{H}_R$ denote the truncation of H_R at level 1,

$$\bar{H}_R = \min\{H_R, 1\}.$$

Since

$$(y+C_1)\log R - y\log(y+1) - \frac{1}{R}y^2$$

is strictly increasing in the interval $[0, R/8]$ we conclude that

$$H_R \geq G+2 > 1 \text{ if either } y > CR/\log R \text{ or } |t| > R/16.$$

Hence $\bar{H}_R = 1$ outside $\mathcal{B}^+_{R/16}$, and we extend $\bar{H}_R = 1$ outside this ball in the whole $\mathbb{R}^2_+$. Finally we define G_R in $\mathbb{R}^2_+$ as

$$G_R(t,y) := \inf_{l\geq 0} \bar{H}_R(t+l, y).$$

Next we collect some key properties of the function G_R.

Lemma 2.1 (Supersolution profile)**.** *Then for all large R we have*

1) $G_R = 1$ outside $\mathcal{B}^+_{R/8} \cup ((-\infty,0]\times[0,R/8])$,

2) $G_R(t,y)$ is nondecreasing in t, and $\partial_t G_R = 0$ outside $\mathcal{B}^+_{R/8}$,

3) $G_R = H_R$ in $\mathcal{B}^+_{R^{1/3}}$ and

$$|G_R - G| \leq C\frac{\log R}{R} \quad \text{in } \mathcal{B}^+_4,$$

4)

$$\Delta G_R + \frac{2(n-1)}{R}\partial_t G_R \leq 0,$$

and on $y=0$:

$$\partial_y G_R < W'(G_R) + \chi_{[-1,1]}\frac{C\log R}{R}.$$

The inequalities in 4) are understood in the viscosity sense.

Notice that by (2.4), property 3) implies that

$$G_R(t,y) \leq G\left(t + C''\frac{\log R}{R}, y\right) \quad \text{in } \mathcal{B}^+_4.$$

Proof. Properties 1) and 2) follow from the definition of G_R since $\bar{H}_R = 1$ outside $\mathcal{B}^+_{R/8}$. We compute

$$\partial_t H_R = G_t + \frac{2tC_0}{R^2}, \tag{2.6}$$

and use (2.4) to conclude that $\partial_t H_R > 0$ if $t \geq -R^{1/2}$. In $\mathcal{B}^+_{R^{1/3}}$ we have

$$|H_R| \leq |G| + \frac{C_0}{R}(R^{1/2} + R^{-1/3}) \leq 1 - cR^{-1/3} + CR^{-1/2} < 1,$$

hence

$$G_R = H_R \quad \text{in } \mathcal{B}^+_{R^{1/3}}.$$

For property 4) we use that G_R is the infimum over a family of left translations of H_R, hence it suffices to show the inequalities for H_R in the region where $H_R < 1$ and $\partial_t H_R > 0$. This means that we can restrict to the region where $1+y \leq CR/\log R$, and $|t| \leq CR^{2/3}$. From the definition (2.5) of H_R we have

$$\Delta H_R = -\frac{C_0}{R}\left(\frac{1}{(1+y)^2} + \frac{1}{1+y}\right) \leq -\frac{C_0}{R}\frac{1}{1+y},$$

and, by (2.6)

$$\partial_t H_R \leq C' \frac{1+y}{1+r^2} + C_0 \frac{r}{R^2}.$$

Since $1+y \leq CR/\log R$ we easily obtain the first inequality in 4) by choosing C_0 large depending on C'. On $y=0$ we have

$$\begin{aligned} \partial_y H_R &= \partial_y G + C_0 \frac{\log R}{R} = W'(G) + C_0 \frac{\log R}{R}, \\ H_R &= G + C_0 C_1 \frac{\log R}{R} + C_0 \frac{t^2}{R^2}. \end{aligned}$$

Then, by (2.3), $W''(g) > c$ outside the interval $[-1,1]$, and we find that

$$W'(H_R) \geq W'(G) + cC_1 C_0 \frac{\log R}{R} > \partial_y H_R,$$

which easily gives the desired conclusion. □

3 Estimates for $\{u=0\}$

In this section we derive properties of the level sets of solutions to

$$\Delta U = 0, \quad \partial_y U = W'(U), \tag{3.1}$$

which are defined in large domains.

In the next lemma we use the functions G_R constructed in the previous section and find axial approximations to (3.1).

Lemma 3.1 (Axial approximations). *Let $G_R:\mathbb{R}^2_+\to(-1,1]$ be the function constructed in Lemma 2.1. Then its axial rotation in $\mathbb{R}^{n+1}$*

$$\Phi_R(x,y):=G_R(|x|-R,y)$$

satisfies

1) $\Phi_R=1$ outside $\mathcal{B}^+_{2R}$, and Φ_R is constant in $\mathcal{B}^+_{R/2}$.

2)

$$\Delta\Phi_R\le 0 \quad in \ \mathbb{R}^{n+1}_+,$$

and

$$\partial_y\Phi_R<W'(\Phi_R), \quad when \ |x|-R\notin[-1,1].$$

3) In the annular region $|(|x|-R,y)|\le R^{\frac{1}{3}}$, we have

$$|\Delta\Phi_R|\le C\frac{1}{R}, \quad |\partial_y\Phi_R-W'(\Phi_R)|\le C\frac{\log R}{R}.$$

Let $\phi_R(x)=\Phi_R(x,0)$ denote the trace of Φ_R on $\{y=0\}$. Notice that ϕ_R is radially increasing, and $\{\phi_R=0\}$ is a sphere which is in a $C\log R/R$-neighborhood of the sphere of radius R.

Proof. We have

$$\begin{aligned}&\Delta\Phi_R(x,y)=\Delta G_R(s,y)+\frac{n-1}{R+s}\partial_s G_R(s,y), \quad s=|x|-R,\\&\partial_y\Phi_R(x,0)=\partial_y G_R(s,0).\end{aligned}$$

The conclusion follows from Lemma 2.1 since $\partial_s G_R=0$ when $|s|\ge R/8$ and $R+s>R/2$ when $|s|<R/8$. □

Definition 3.1. We denote by $\Phi_{R,z}$ the translation of Φ_R by z i.e.,

$$\Phi_{R,z}(x,y):=\Phi_R(x-z,y)=G_R(|x-z|-R,y).$$

Sliding the graph of Φ_R:

Assume that u is less than ϕ_{R,x_0} in $B_{2R}(x_0)$. By the maximum principle we obtain that $U<\Phi_{R,z}$ with $z=x_0$ in $\mathcal{B}_{2R}(x_0,0)$ (and therefore globally.) We translate the function Φ_R above by moving continuously the center z, and let's assume that it touches U by above, say for simplicity when $z=0$, i.e., the strict inequality becomes equality for some contact point (x^*,y^*). From Lemma 3.1 we know that Φ_R is a strict supersolution away from $\{y=0\}$, and moreover the contact point must satisfy $y^*=0$, $|x^*|-R\in[-1,1]$, that is it belongs to the annular region $B_{R+1}\setminus B_{R-1}$ in the n-dimensional subspace $\{y=0\}$.

Lemma 3.2 (Estimates near a contact point)**.** *Assume that the graph of* Φ_R *touches by above the graph of* U *at a point* $(x^*,0,u(x^*))$ *with* $x^* \in B_{R+1} \setminus B_{R-1}$.

Then in $B_2(x^*,0)$ *the level set* $\{u=0\}$ *stays in a* $C\log R/R$ *neighborhood of the sphere* $\partial B_R = \{|x|=R\}$, *and*

$$\|u-\phi_R\|_{C^{1,1}(B_2(x^*))} \le C\frac{\log R}{R}.$$

Proof. Assume for simplicity that x^* is on the positive x_n axis, thus $|x^*-Re_n| \le 1$. By Lemma 3.1 we have

$$U \le \Phi_R \le G\left(x_n - R + C\frac{\log R}{R}, y\right) =: V \quad \text{in } \mathcal{B}_3(Re_n).$$

Both U and V solve the same equation (3.1), and

$$(V-U)(x^*,0) \le C\frac{\log R}{R}.$$

Since $V-U \ge 0$ satisfies

$$\Delta(V-U)=0, \quad \partial_y(V-U)=b(x)(V-U),$$
$$b(x) := \int_0^1 W''(tu(x)+(1-t)v(x))dt,$$

we obtain

$$|V-U| \le C\frac{\log R}{R} \quad \text{in } \mathcal{B}_{5/2}(Re_n),$$

from the Harnack inequality with Neumann boundary condition. Moreover since b has bounded $C^{1,\alpha}$ norm, we obtain that $U-V \in C_x^{2,\alpha}$ for some $\alpha>0$, and

$$\|U-V\|_{C^{1,1}(\mathcal{B}_2(Re_n))} \le C\frac{\log R}{R},$$

by local Schauder estimates. This easily implies the lemma. □

Remark 3.1. If instead of Φ_R being tangent by above to U, we only assume

$$\Phi_R \ge U \quad \text{and} \quad (\phi_R - u)(x^*) =: a \le c$$

at some $x^* \in B_{R+1} \setminus B_{R-1}$ then the Harnack inequality above gives

$$ca - C\frac{\log R}{R} \le \Phi_R - U \le C\left(a + \frac{\log R}{R}\right) \quad \text{in } \mathcal{B}_2(x^*,0),$$

for some C large, universal.

Lemma 3.3. *Assume that*

a) $B_R(-Re_n)\subset\{u<0\}$ *is tangent to* $\{u=0\}$ *at* 0.

b) there is $x_0\in B_{R/2}(-Re_n)$ *such that* $u(x_0)\le -1+c$ *for some* $c>0$ *small.*

Denote by D *the set*

$$D:=\{u<0\}\setminus B_R(-Re_n).$$

Then

$$\int_{B_1^c}\frac{\chi_D(x)}{|x|^{n+1}}dx\le C\frac{\log R}{R}, \tag{3.2}$$

and

$$\{u=0\}\cap B_{R^\sigma}\subset\{|x_n|\le R^{-3/4}\},$$

for some $\sigma>0$ *small, universal.*

We remark that in (3.2) we can integrate over whole $\mathbb{R}^n$ instead of B_1^c since, by Lemma 3.2, the curvatures of ∂D in B_1 are bounded by $C\log R/R$.

Proof. First we claim that

$$U\le\Phi_{R/2,t_0e_n},\quad \text{with }\ t_0=-\frac{R}{2}-K\frac{\log R}{R}, \tag{3.3}$$

for some K large universal.

Let's assume first that $\Phi_{R/2,te_n}\ge U$ when $t=-R$. We want to show that this inequality remains valid as we increase t from $-R$ till t_0. By Lemma 3.1, the first contact point between the graphs of U and Φ_{R,te_n} can occur only on $y=0$ and, by Lemma 3.2 near this contact point the $\{u=0\}$ and $|x+te_n|=R/2$ must be at most $C\log R/R$ apart. This is not possible if K is chosen sufficiently large.

To prove that $\Phi_{R/2,-Re_n}\ge U$, one can argue similarly by using hypothesis b) and looking at the continuous family $\Phi_{r,-Re_n}$ and then increase r from C to $R/2$. This proves the claim (3.3).

We write $\Phi=\Phi_{R/2,t_0e_n}$ for simplicity of notation, and by ϕ the trace of Φ on $y=0$. We have $\Phi\ge U$, and therefore $\phi\ge u$, and

$$\Delta^{1/2}\phi(0)\le W'(\phi(0))+C\frac{\log R}{R}.$$

Using that

$$|\phi(0)|\le C(K)\log R/R,\quad u(0)=0,$$

together with the equation for $\Delta^{1/2}u$ at 0 we obtain that

$$\Delta^{1/2}(\phi-u)(0)\le C(K)\frac{\log R}{R}.$$

Since $(\phi-u)(0)$ and $\|\phi-u\|_{C^{1,1}(B_1)}$ (by Lemma 3.2) are bounded by $C\log R/R$, we use the integral representation for $\Delta^{1/2}$ and obtain

$$\int_{B_1^c}\frac{\phi-u}{|x|^{n+1}}dx\le C\frac{\log R}{R}. \tag{3.4}$$

Next we show that we can replace $\phi-u$ in the integral above by $\chi_{\tilde{D}}$ where

$$\tilde{D}:=\{u<0\}\setminus\{\phi<0\}\supset D.$$

For this it suffices to show that for any unit ball $B_1(z)$ with center $z\in\tilde{D}$ we have

$$c_1\int_{B_1(z)}\chi_{\tilde{D}}dx\le\int_{B_1(z)}(\phi-u)+C_1\frac{\log R}{R}dx,$$

for some $c_1>0$ small, and C_1 large universal.

Indeed, let $a=(\phi-u)(z)$. If $a>c$ then, by the Lipschitz continuity of $\phi-u$, the right hand side above is bounded below by a universal constant and the inequality is obvious. If $a<c$ then we use Remark 3.1 and conclude that

$$|\tilde{D}\cap B_1(z)|\le C\left(a+\frac{\log R}{R}\right),$$

and

$$\phi-u\ge ca-C\frac{\log R}{R}\quad\text{in } B_1(z),$$

which gives the desired inequality by choosing C_1 sufficiently large.

Next we show that

$$\varphi-u\le R^{-4/5}\quad\text{ in } B_{R^\sigma}, \tag{3.5}$$

for some small $\sigma>0$. Assume by contradiction that

$$(\varphi-u)(z)>R^{-4/5}\quad\text{for some } z\in B_{R^\sigma}.$$

Let $V:=\Phi-U\ge 0$. We have $V(z,0)>R^{-3/4}$, and by part 3) of Lemma 3.1,

$$|\Delta V|\le\frac{C}{R},\quad|\partial_y V|\le CV+C\frac{\log R}{R}\quad\text{in } \mathcal{B}_2(z).$$

By Harnack inequality we obtain

$$V\ge cR^{-4/5}\quad\text{in } \mathcal{B}_{3/2}(z).$$

This means that the left hand side in (3.4) is greater than $cR^{-4/5}R^{-(n+1)\sigma}$ and we reach a contradiction if we choose σ small depending only on n. Hence the claim (3.5) is proved. This implies that in B_{R^σ}, the set $\{u=0\}$ is in a $CR^{-4/5}$ neighborhood of the 0 level set of ϕ which gives the desired conclusion. □

Remark 3.2. In the proof above we obtain

$$|\Phi - U| \le R^{-3/4} \quad \text{in } \mathcal{B}_{R^\sigma}. \tag{3.6}$$

Indeed, in $\mathcal{B}_{R^\sigma}$ we have

$$V \ge 0, \quad |\Delta V| \le \frac{C}{R}, \quad |\partial_y V| \le C R^{-4/5},$$

where in the last inequality we used that on $y=0$, $V \le R^{-4/5}$ by (3.5). Now (3.6) follows from Harnack inequality provided that σ is sufficiently small.

As a consequence of Lemma 3.3 we obtain

Corollary 3.1. Assume that $B_R(-Re_n) \subset \{u<0\}$ is tangent to $\{u=0\}$ at 0. Then in the cylinder $\{|x'| \le R^{\frac{\sigma}{6}}\}$ the set $\{u=0\}$ cannot lie above the surface

$$x_n = \frac{1}{R}\left(\Lambda (x\cdot e')^2 - |x'|^2\right),$$

where e' is a unit direction with $e' \cdot e_n = 0$ and Λ is a large universal constant.

Proof. Indeed, otherwise the integral in (3.2) is greater than

$$\int_1^{R^{\sigma/6}} c\frac{\Lambda}{R}\frac{r^n}{r^{n+1}}dr \ge c(\sigma)\Lambda\frac{\log R}{R},$$

and we reach a contradiction if Λ is chosen sufficiently large. □

4 A mean curvature estimate for $\{u=0\}$

In this section we refine some of the results of last section and we estimate the mean curvature of a surface that touches $\{u=0\}$ by below at 0, in a neighborhood of size $l \ge R^{1/3}$.

Proposition 4.1. Fix $\delta>0$ small and let R, l be large with $l \in [R^{1/3}, \delta^3 R]$, and let θ denote

$$\theta := l^2 R^{-1}.$$

Assume that in the ball B_l the surface

$$\Gamma := \left\{ x_n = \sum_1^{n-1} \frac{a_i}{2} x_i^2 + b'\cdot x' + b_0 \right\},$$

with

$$|a_i| \le \delta R^{-1}, \quad |b'| \le \delta l R^{-1}, \quad |b_0| \le \theta,$$

is tangent to $\{u=0\}$ at $b_0 e_n$.

Assume further that $\{u<0\}$ contains the two balls $B_R(-t_0 e_n)$ and $B_{R_m}(-t_m e_n)$ of radii R and R_m and passing through $-\theta e_n$ and respectively $-\theta_m e_n$ with

$$t_0=\theta+R, \quad t_m=\theta_m+R_m, \quad R_m:=2^{\frac{1}{2}m}R, \quad \theta_m:=2^{\frac{3}{2}m}\theta.$$

Then

$$\sum_1^{n-1} a_i \le \delta^4 R^{-1},$$

if $m=m(\delta)$ is chosen sufficiently large depending only on δ and the universal constants.

Proof. First we claim that at each point $x_0\in\Gamma\cap B_{l/3}$ we have a tangent ball of radius $\frac{1}{16}R$ by below which is included in the set $\{u<0\}$.

Indeed, the bounds on $|a_i|$, $|b_i|$ imply that at x_0, Γ has a quadratic polynomial

$$x_n=-\frac{8}{R}|x'-z_0'|^2+c_{z_0}$$

tangent by below, with

$$|x_0'-z_0'|\le C\delta l, \quad c_{z_0}\le 2\theta.$$

It is straightforward to check that the quadratic surface above lies inside the ball $B_R(t_0 e_n)$ in the region $B_R\setminus B_l$, and our claim easily follows.

As in the proof of Lemma 3.2, $B_{R_m}(t_m e_n)\subset\{u<0\}$ gives the bound

$$U\le \Phi_{R_m/2,te_n},$$

as long as the ball $B_{R_m/2}(te_n)$ lies inside the ball

$$\tilde{B}:=B_{\tilde{R}}(t_m e_n), \quad \tilde{R}:=R_m-C\frac{\log R_m}{R_m},$$

for some C large, universal. This gives the bound

$$U\le G_{R_m/2}(d_m+C\log R_m/R_m,y), \tag{4.1}$$

where d_m denotes the signed distance to the sphere $\partial B_{R_m}(-t_m e_n)$, with $d_m>0$ outside the ball.

Similarly, we use that at each point in $\Gamma\cap B_{l/3}$ the tangent ball of radius $\frac{1}{16}R$ by below is included in $\{u<0\}$ and we obtain

$$U\le G_{R/32}(d_\Gamma+C\log R/R) \quad \text{in } \mathcal{B}_{l/4}, \tag{4.2}$$

where d_Γ represents the signed distance to the the surface Γ.

We assume by contradiction that the conclusion is not satisfied i.e.,

$$\sum a_i>\delta^4 R^{-1}.$$

Since on $\Gamma \cap B_l$ the slope of Γ (viewed as a graph in the e_n direction) is bounded by $C(n)\delta l R^{-1} \le \delta^2$ and $|a_i| \le \delta R^{-1}$ we obtain

$$H_\Gamma \ge \sum a_i - C\delta^4 \max|a_i| \ge \frac{1}{2}\delta^4 R^{-1},$$

where H_Γ represents the mean curvature of Γ. Moreover, the curvatures of Γ are bounded by $2\delta R^{-1}$, which easily gives that in B_l all parallel surfaces to Γ satisfy a similar mean curvature bound:

$$H_\Gamma(x) \ge \sum a_i - Cl(\delta R^{-1})^2 \ge \frac{1}{2}\delta^4 R^{-1}, \quad \forall x \in B_l, \tag{4.3}$$

where we have used the hypothesis $lR^{-1} \le \delta^3$. Here $H_\Gamma(x)$ denotes the mean curvature of the parallel surface to Γ passing through x.

Next we use (4.3) to construct a supersolution with 0 level set sufficiently close to Γ. Then we make use of (4.1), (4.2) and reach a contradiction by showing that this supersolution touches U by above at an interior point.

For the construction of the supersolution we first introduce a 2D profile in the (t,y) variables which is a perturbation of G. It is similar to the profile H_R defined in (2.5). Precisely we define H^* in $\mathbb{R}^2_+$ as

$$H^*(t,y) := G + \frac{c(\delta)}{R} h(t,y) \tag{4.4}$$

with

$$h(t,y) := c_1 \varphi(2r) y \log r + \varphi(r) \frac{t^2 - y^2}{r}, \tag{4.5}$$

where $r = |(t,y)|$ is the distance from (t,y) to the origin, and φ is a cutoff function with $\varphi = 0$ in $[0,1]$ and $\varphi = 1$ in $[2,\infty)$. The constant c_1 is small, universal, and the constant $c(\delta) > 0$ depends also on δ will be made precise below. Outside $\mathcal{B}_4^+$, the function h has the property that

a) Δh is homogenous of degree -1 and

b) on $y=0$, $h=|t|$ and $h_y = c_1 \log|t|$.

The following properties hold provided that c_1 is sufficiently small:

1) h is superharmonic in an angular region near the t axis

$$\Delta h < 0 \quad \text{in the region } \{y < |t|/2\} \setminus \mathcal{B}_4^+,$$

2) Outside this region we have the bounds (see (2.4))

$$\Delta h \le C \min\{1, r^{-1}\}, \quad \partial_t H^* \ge c \min\{1, r^{-1}\},$$

3) On $y=0$, we have $h(t,0) \ge 0$ and

$$h_y = h = 0 \text{ in } [-1,1] \quad \text{and} \quad h_y \le ch \text{ outside } [-1,1],$$

for some $c>0$ universal, smaller than the minimum of W'' outside the interval $[g(-1),g(1)]$ (see (2.3)).

These properties imply that in the region where $\partial_t H^* \geq 0$ we have

$$\Delta H^* - \frac{\delta^4}{8}\frac{1}{R}\partial_t H^* \leq 0,$$

provided that $c(\delta)$ is chosen sufficiently small, and

$$\partial_y H^* \leq W'(H^*) \quad \text{on } \; y=0.$$

Next we modify the 2D profile H^* by cutting at level 1 and making it increasing in the t variable. We define G^* as the infimum over left translations in a similar fashion as we did for H_R. Precisely, we define

$$\bar{H}^* := \min\{H^*,1\}, \quad G^* := \inf_{l\geq 0} \bar{H}^*(t+l,y),$$

and then $\partial_t G^* \geq 0$ by construction. Moreover, G^* satisfies the inequalities above (in the viscosity sense):

$$\Delta G^* - \frac{\delta^4}{8}\frac{1}{R}\partial_t G^* \leq 0, \tag{4.6}$$

and

$$\partial_y G^* \leq W'(G^*) \quad \text{on } \; y=0. \tag{4.7}$$

In the next lemma we compare the profiles G^* with appropriate translations of $G_{R/32}$ respectively $G_{R_m/2}$.

Lemma 4.1. *We have the following inequalities:*

a) on $y=0$

$$G^*(t,0) \geq G_{R/32}(t-R^{-1/2},0), \tag{4.8a}$$

$$G^*(t,0) \geq G_{R/32}(t+R^{-1/2},0), \quad \text{if } \; |t|>R^{1/4}. \tag{4.8b}$$

b) in $\mathbb{R}^2_+$ *we have*

$$G^*(t,y) \geq G_{R/32}(t+K\log R/R,y) - C(K)\frac{\log R}{R}(y+1), \tag{4.9a}$$

$$G^*(t,y) \geq G_{R_m/2}(t+2\theta_m,y) + c_1(\delta)\frac{\log R}{R}y, \quad \text{if } \; y \geq l(\log R)^{-\frac{1}{3}}, \tag{4.9b}$$

provided that $k=k(\delta)$ *is chosen sufficiently large.*

Proof. It suffices to show the inequalities for H^* and H_R in the regions where $\{\partial_t H^* > 0\} \cap \{H^* < 1\}$ and then the desired results for G^* and G_R follow by taking the infimum over left translations. First we check that

$$\{\partial_t H^* > 0\} \cap \{H^* < 1\} \subset \{r \leq R(\log R)^{-1/3}\}. \tag{4.10}$$

We notice from (4.4) that

$$h(t,y) \geq c(y\log r + r) \quad \text{outside } \mathcal{B}_C^+. \tag{4.11}$$

This means that if $y > CR/\log R$ then $H^* > 1$. In the two regions where $y < CR/\log R$ and $r > R(\log R)^{-1/3}$ we have

a) either $t > r/2$ and then we easily obtain $H^* > 1$ by using (see (2.4))

$$G \geq 1 - C\frac{1+y}{r},$$

b) or $t < -r/2$ and we obtain $H_t^* < 0$ by using

$$G_t \leq C(1+y)r^{-2}, \quad \text{and} \quad h_t \leq -c,$$

and (4.10) is proved.

To prove a) we have (see (4.5), (2.5))

$$\begin{aligned} &H^*(t,0) \geq g(t) + \Big(1 - \chi_{[-2,2]}\Big)\frac{c(\delta)|t|}{R}, \\ &H_{R/32}(t \pm R^{-1/2}, 0) \leq g(t \pm R^{-1/2}) + C\frac{\log R}{R} + C\left(\frac{t}{R}\right)^2. \end{aligned}$$

The two inequalities follow easily since $|t|/R = o(1)$ by (4.10), and by (2.2) we have

$$g(t - R^{-1/2}, 0) \leq g(t) - c\frac{R^{-1/2}}{1+t^2}, \quad \text{and} \quad g(t + R^{-1/2}, 0) \leq g(t) + C\frac{R^{-1/2}}{1+t^2}.$$

For part b) we estimate translations of H_R as

$$H_R(t+\sigma, y) \leq G(t+\sigma, y) + C\frac{\log R}{R}(y+1) + C\frac{t^2+\sigma^2}{R^2}$$

and using that $G_t \leq C/(y+1)$ we have

$$H_R(t+\sigma, y) \leq G(t,y) + C\frac{\sigma}{y+1} + C\frac{\log R}{R}(y+1) + C\left(\frac{t}{R}\right)^2.$$

The third inequality is easily verified by taking $\sigma = K\log R/R$, $C(K)$ sufficiently large and then using (4.11) to estimate H^*.

Finally, for (4.9b) we take $\sigma = 2\theta_m = 2^{\frac{3}{2}m+1}\theta$ and replace R with $\frac{1}{2}R_m = 2^{\frac{1}{2}m-1}R$. in the inequality above.

We restrict to the region $y \geq l(\log R)^{-1/3} \geq R^{1/4}$, thus we have $r \geq R^{1/4}$. We first choose m large such that

$$\frac{c}{3}\frac{\log r}{R}y \geq C\frac{\log R_m}{R_m}(y+1),$$

and then

$$\frac{c}{3}\frac{\log r}{R}y \geq C\frac{2\theta_m}{y+1},$$

for all large R's, and the lemma is proved. □

In the ball $\mathcal{B}_{l/4}$ we define the function

$$\Psi := G^*(d_{\tilde{\Gamma}}, y),$$

where $d_{\tilde{\Gamma}}$ is the signed distance to the surface

$$\tilde{\Gamma} = \left\{ x_n = \sum_1^{n-1} \frac{a_i}{2} x_i^2 - \frac{\delta^4}{4nR}|x'|^2 + b' \cdot x' \right\}.$$

From the properties of G^* we find that Ψ is a supersolution, which is increasing in the e_n direction. Indeed, at a point $(x,y) \in \mathcal{B}_{l/4}$ we have (as in (4.3))

$$H_{\tilde{\Gamma}}(x) > \frac{1}{8}\delta^4 R^{-1},$$

and we compute (see (4.6), (4.7))

$$\Delta\Psi(x,y) = \Delta G^*(s,y) - H_{\tilde{\Gamma}}(x)\partial_s G^*(s,y) < 0,$$

where $s = d_{\tilde{\Gamma}}$. Also, on $\{y=0\}$

$$\partial_y \Psi = \partial_y G^*(s,0) \leq W'(G^*) = W'(\Psi).$$

We claim that on $\{y=0\}$ we have

$$\Psi > U \quad \text{outside } B_{l/8}. \tag{4.12}$$

To prove this, in view of (4.2), it suffices to show that

$$G^*(d_{\tilde{\Gamma}}, 0) > G_{R/32}(d_1, 0), \quad d_1 := d_\Gamma + C\log R/R.$$

Indeed in $B_{l/4} \setminus B_{l/8}$ we either have
a) $d_{\tilde{\Gamma}} > d_\Gamma + cl^2R^{-1} > d_1 + R^{-1/2}$ or,
b) $|d_{\tilde{\Gamma}}| \geq l/16 > R^{1/4}$ and $d_{\tilde{\Gamma}} \geq d_\Gamma > d_1 - R^{-1/2}$.

The claim follows then from part a) of Lemma 4.1 above.

Moreover, in $\mathcal{B}_{l/4}$ we have $d_{\tilde{\Gamma}} \geq d_\Gamma$ hence by (4.9a)

$$\Psi > U - C\frac{\log R}{R}(y+1).$$

Since

$$d_{\tilde{\Gamma}} + 2\theta_m \geq d_m + C\log R_m / R_m,$$

(recall (4.1) for the definition of d_k) we find by (4.9b), (4.1), that

$$\Psi > U + c\frac{\log R}{R} y \quad \text{if } y > l(\log R)^{-\frac{1}{3}} =: l\varepsilon_R.$$

Thus on $\partial\mathcal{B}_{l/4}$ we have

$$\Psi > U - C\gamma \quad \text{if } y < l(2\varepsilon_R), \quad \text{and} \quad \Psi > U + c\gamma \quad \text{otherwise},$$

where

$$\gamma := \frac{\log R}{R}(l\varepsilon_R).$$

Next we translate the graph of Ψ in the $-e_n$ direction till, on $y=0$ it becomes tangent by above to the graph of U. Indeed, since $\Psi(0)=U(0)$ and $\Psi > U$ outside $B_{l/8}\times\{0\}$, we can translate Ψ so that $\Psi_0(X) := \Psi(X + t_1 e_n)$, for some $t_1 \geq 0$, becomes tangent by above to U on $y=0$ at some point $(x^*,0) \in B_{l/8}$.

Now we see that $V := \Psi_0 - U \geq \Psi - U$ satisfies in $B_{l/4}$:

$$\Delta V \leq 0, \quad V \geq 0 \quad \text{on } \{y=0\}, \quad V(x^*,0) = 0,$$

and on $\partial\mathcal{B}_{l/4}$,

$$V \geq -C\gamma \quad \text{if } y < l(2\varepsilon_R), \quad \text{and} \quad V \geq c\gamma \quad \text{otherwise}.$$

Since ε_R can be taken arbitrarily small we find $V \geq 0$ in $\mathcal{B}_{3l/8}$ and therefore we also obtain $V_y(x^*,0) > 0$. This means

$$U_y < \partial_y \Psi_0 \leq W'(\Psi_0) = W'(U) \quad \text{at} \quad (x^*,0),$$

and we reached a contradiction. □

5 Harnack inequality

In this section we prove a Harnack inequality property for flat level sets, see Proposition 5.1 below. We will make use of Lemma 3.3 and Corollary 3.1 together with a standard Γ-convergence result for minimizers, see Lemma 5.2.

Notation: We denote by $\mathcal{C}(l,\theta)$ the cylinder

$$\mathcal{C}(l,\theta) := \{|x'| \leq l, \ |x_n| \leq \theta\}.$$

Proposition 5.1 (Harnack inequality for minimizers)**.** Let U be a minimizer of $\mathcal{J}$ and assume that

$$\{u=0\} \cap \mathcal{C}(l,l) \subset \mathcal{C}(l,\theta),$$

and that the balls of radius $C' l^2 \theta^{-1}$ (with C' universal) which are tangent to $\mathcal{C}(l,\theta)$ at $\pm\theta e_n$ by below and above are included in $\{u<0\}$ respectively $\{u>0\}$.

Given $\theta_0>0$, there exist $\varepsilon_0(\theta_0)>0$ depending on θ_0, such that if

$$\theta l^{-1}\leq\varepsilon_0(\theta_0),\quad \theta_0\leq\theta, \tag{5.1}$$

then

$$\{u=0\}\cap\mathcal{C}\Big(\frac{l}{4},\frac{l}{4}\Big)$$

is either included in $\{x_n\leq(1-\omega_0)\theta\}$ or in $\{x_n\geq-(1-\omega_0)\theta\}$, with $\omega_0>0$ small universal.

After a translation in the e_n direction, the conclusion can be stated as

$$\{u=0\}\cap\mathcal{C}(\bar{l},\bar{l})\subset\mathcal{C}(\bar{l},\bar{\theta})\quad \text{with}\quad \bar{l}:=\frac{l}{4},\quad \bar{\theta}:=\Big(1-\frac{\omega_0}{2}\Big)\theta.$$

We remark that if (5.1) is satisfied again for $\bar{\theta}$, $\bar{l}$, then we can apply Proposition 5.1 again since the hypothesis that the tangent ball of radius $C'\bar{l}^2\bar{\theta}^{-1}$ tangent by below to $\mathcal{C}(\bar{l},\bar{\theta})$ is included in $\{u<0\}$ is clearly satisfied.

Recall that $G(t,y)$ has the property

$$\mathcal{J}(G,B_R^+)=C^*\log R+\mathcal{O}(1), \tag{5.2}$$

for some constant $C^*>0$. Moreover, G is a minimizer of $\mathcal{J}$ in B_R^+ among functions with values between -1 and 1 which agree with G on $\partial B_R^+\setminus\{y=0\}$.

Before we proceed with the proof of Proposition 5.1 we need some energy bounds for functions defined in half-squares

$$Q_l:=[-l,l]\times[0,l].$$

Lemma 5.1. *a) Assume that V is Lipschitz, defined in $Q_{l+1}\subset\mathbb{R}^2_+$, $|V|\leq 1$ and*

$$V(t,0)\leq-1+\gamma^2 \text{ if } t\leq-\tfrac{l}{2},\quad \text{and}\quad V(t,0)\geq 1-\gamma^2\quad \text{if } t\geq\tfrac{l}{2}, \tag{5.3}$$

for some small γ. Then for all sufficiently large l we have

$$\mathcal{J}(V,Q_l)\geq(C^*-\gamma)\log l. \tag{5.4}$$

b) Moreover, if we assume that there exist two points s_1, s_2 in $[-l/2.l/2]$ with $|s_1-s_2|\geq\theta_0$, such that

$$|V(t+s_i,y)-G(t,y)|\leq c(\theta_0)\quad \text{in}\ \mathcal{B}^+_{l^\sigma},$$

for some given $\sigma>0$ small and $c(\theta_0)$ sufficiently small, then

$$\mathcal{J}(V,Q_l)\geq(C^*+c_0(\sigma))\log l, \tag{5.5}$$

for some constant $c_0(\sigma)>0$.

The proof of Lemma 5.1 is postponed till the end of this section.

Proof of Proposition 5.1. First we remark that $l\geq\theta_0\varepsilon_0^{-1}\to\infty$ as $\varepsilon_0\to 0$.

Let A be the rescaling of the 0 level set of u given by

$$(x',x_n)\in\{u=0\}\mapsto(z',z_n)\in A,$$
$$z=Tx,\quad (z',z_n)=T(x',x_n):=(x'l^{-1},x_n\theta^{-1}).$$

Our hypothesis is that $A\subset\mathcal{C}(1,1)$ and we want to show that in the cylinder $|z'|\leq\frac{1}{4}$ the set A is included either in $z_n\leq 1-\omega_0$ or in $z_n\geq -1+\omega_0$.

We view A as a multivalued graph over $z'\in B_1'$.

Let us assume that we touch A by below at a point $z_0\in A$ with the graph of a quadratic polynomial $P_{p'}^{\mu}$ of opening $-\mu$ and vertex $p'\in B_{1/3}'$

$$z_n=P_{p'}^{\mu}(z'):=-\frac{\mu}{2}|z'-p'|^2+c_{p'}\quad \text{for some constant } \ c_{p'}\leq -1+\frac{\mu}{8},$$

and $\mu\in[\omega_0,1]$ with ω_0 a small universal constant to be specified later.

We claim that Lemma 3.3 and Corollary 3.1 imply that A satisfies the following two properties:

a) A contains a graph which is fully included in the cylinder $z_0+\mathcal{C}(l^{\sigma-1},2\mu)$.

b) A cannot be touched at z_0 in a $B_{r_0}(z_0)$ neighborhood with

$$r_0:=l^{\sigma/2-1},$$

by the graph

$$\Big\{z_n=P_{p'}^{\mu}+4\mu\Lambda((z'-z_0')\cdot e')^2,\quad |z'-z_0'|\leq r_0\Big\},$$

with $e'\in\mathbb{R}^{n-1}$ a unit direction. Here σ and Λ represent the universal constants that appear in Lemma 3.2 and Corollary 3.1.

Indeed, the restrictions on p' and $c_{p'}$ imply that $|z_0'|\leq 5/6$ and $z_0\cdot e_n\leq -1+\frac{\mu}{8}$. The corresponding point

$$x_0:=T^{-1}z_0,\ \text{then satisfies}\ \ |x_0'|\leq 5l/6,\quad x_0\cdot e_n\leq(-1+\tfrac{\mu}{8})\theta.$$

Moreover, if the constant C' in our hypothesis is chosen large depending on ω_0, then the ball of radius

$$q=l^2(2\mu\theta)^{-1},$$

which is tangent to $\{u=0\}$ at x_0 by below is globally included in $\{u<0\}$. The outer normal ν to this ball at x_0 satisfies $|\nu-e_n|\leq\mu\theta l^{-1}$ and Lemma 3.3 implies that

$$\{u=0\}\cap B_{q^\sigma}(x_0)\subset\{|(x-x_0)\cdot\nu|\leq q^{-3/4}\}. \tag{5.6}$$

We use that $q\geq cl\varepsilon_0^{-1}\geq Cl$ provided that ε_0 is chosen small, and

$$q^{-3/4}\leq l^{-3/4}\leq\omega_0\theta_0\leq\mu\theta,$$

and property a) above follows by rescaling back (5.6) to the z variable.

Property b) holds since otherwise, as above, we end up at the point x_0 with a surface as in Corollary 3.1 tangent to $\{u=0\}$ by below in a $\frac{1}{2}l^{\sigma/2}$- neighborhood of x_0. This neighborhood includes $B_{q^{\sigma/6}}(x_0)$ since $l^3 \geq Cl^2 \geq Cq$ and we reach a contradiction and the claim is proved.

By Remark 3.2 we obtain in (5.6) also information on the whole profile U

$$|U - G((x-x_0)\cdot\nu, y)| \leq q^{-3/4} \quad \text{in } \mathcal{B}^+_{q^\sigma}(x_0).$$

This implies that for each $x \in B_{l^\sigma}(x_0) \cap \{u=0\}$ we have

$$|U(x+te_n, y) - G(t,y)| \leq Cl^{-3/4} + C\theta l^{-1} \leq \rho(\varepsilon_0), \tag{5.7}$$

if $|(t,y)| \leq l^\sigma$, and $\rho(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$. In the inequality above, we used

$$|(x+te_n-x_0)\cdot\nu - t| \leq |(x-x_0)\cdot\nu| + |\nu - e_n||t| \leq q^{-3/4} + \theta l^{-1}|t|,$$

and (2.4).

Properties a) and b) above state that A satisfies the hypotheses for the general version of Weak Harnack Inequality proved in [3]. Indeed, by property b) the set A cannot be touched by below with the family of surfaces $\overline{\mathcal{P}}^{\mu}_{8\Lambda}(r_0)$, $\mu \in [\omega_0, 1]$, in an neighborhood that contains at least a ball of radius r_0 around the contact point. On the other hand, by property a) Harnack inequality already holds in a $C^*(n,\Lambda) r_0 \leq l^{\sigma-1}$ neighborhood of a contact point, with $C^*(n,\Lambda)$ the universal constant depending only on n and Λ which appears in Theorem 2.4 in [3]. Now we can apply the Theorem 2.4 of [3] and conclude that if

$$A \cap \{|z'| \leq 1/4,\ z_n \leq -1+\omega_0\} \neq \emptyset,$$

then A contains a graph $\underline{A} \subset A$ with

$$\underline{A} \subset B'_{1/4} \times [-1, -1+K\omega_0],$$

with $K = K(n,\Lambda)$ universal, such that

$$\mathcal{H}^{n-1}(\pi_n(\underline{A})) \geq \Big(1 - \frac{1}{4}\Big) \mathcal{H}^{n-1}(B'_{1/4}), \tag{5.8}$$

where π_n denotes the projection in the z' variable.

We choose ω_0 small, depending on K such that $K\omega_0 \leq 1/2$ hence

$$\underline{A} \subset \Big\{ z_n \leq -\frac{1}{2} \Big\}.$$

Similarly, if in the cylinder $z' \in B'_{1/4}$ the set A intersects $z_n \geq 1-\omega_0$, then we can find a graph $\overline{A} \subset A \cap \{z_n \geq 1/2\}$ which satisfies (5.8) as well.

We will reach a contradiction by estimating the energy $\mathcal{J}(U,A_{l/2})$ where

$$A_l := \mathcal{C}(l,l)\times[0,l]\subset\mathbb{R}^{n+1}.$$

Notice that in $\mathcal{C}(\frac{l}{2},\frac{l}{2})$ the function $U(x,0)$ is sufficiently close to ±1 away from a thin strip around $x_n=0$. Indeed, we can use barrier functions as in Lemma 3.2 (see (3.3)) and bound U by above and below in terms of the functions $G_{l/2}(x_n\pm\theta,y)$. This implies that, for any constant γ small, we have

$$|U(x,0)|\geq 1-\gamma^2 \quad \text{in } \mathcal{C}(\tfrac{l}{2},\tfrac{l}{2}) \quad \text{if } |x_n|\geq C(\gamma)+\theta. \tag{5.9}$$

For each $x'\in B'_{l/2}$ we denote by $Q_l(x')$ the 2D half square of size l in the (x_n,y)-variables centered at $(x',0)\in\mathbb{R}^n$ as

$$Q_l(x') := \{(x',t,y)\,|\,|t|\leq l,\ y\in[0,l]\}.$$

Now we can apply Lemma 5.1 part a) and obtain

$$\mathcal{J}(U(x',\cdot),Q_{l/2}(x'))\geq (C^*-\gamma)\log l.$$

On the other hand, if

$$x'l^{-1}=z'\in\pi_n(\overline{A})\cap\pi_n(\underline{A}),$$

then by (5.7), we satisfy the hypotheses of part b) of Lemma 5.1 and obtain

$$\mathcal{J}(U(x',\cdot),Q_{l/2}(x'))\geq (C^*+c_0(\sigma))\log l.$$

In conclusion, after integrating in $x'\in B'_{l/2}$ the inequalities above, and using (5.8) for $\overline{A}$, $\underline{A}$, we obtain that

$$\mathcal{J}(U,A_{l/2})\geq (C^*+c_1)\log l\,\mathcal{H}^{n-1}(B'_{l/2}),$$

for some $c_1>0$ universal, provided that we choose the constant γ sufficiently small.

This contradicts Lemma 5.2 below if ε_0 is sufficiently small.

Lemma 5.2.

$$\mathcal{J}(U,A_{l/2})\leq C^*\log l\left(\mathcal{H}^{n-1}(B'_{l/2})+\eta(\varepsilon_0)l^{n-1}\right). \tag{5.10}$$

with $\eta(\varepsilon_0)\to 0$ as $\varepsilon_0\to 0$.

Proof. We interpolate between U and $V(x,y):=G(x_n,y)$ as

$$H=(1-\varphi)U+\varphi V.$$

Here φ is a cutoff Lipschitz function such that $\varphi=0$ outside $A_{l/2}$, $\varphi=1$ in $\mathcal{R}$ and $|\nabla\varphi|\leq 8/(1+y)$ in $A_{l/2}\setminus\mathcal{R}$, where $\mathcal{R}$ is the cone

$$\mathcal{R} := \{(x,y)\,|\max\{|x'|,|x_n|\}\leq l/2-1-2y\}.$$

By minimality of U we have

$$\mathcal{J}(U,A_{l/2})\leq\mathcal{J}(H,A_{l/2})=\mathcal{J}(V,\mathcal{R})+\mathcal{J}(H,A_{l/2}\setminus\mathcal{R}).$$

By (5.2),

$$\mathcal{J}(V,\mathcal{R})\leq\mathcal{J}(V,A_{l/2})\leq(C^*\log l+\mathcal{O}(1))\mathcal{H}^{n-1}(B'_{l/2}),$$

and we need to show that

$$\mathcal{J}(H,A_{l/2}\setminus\mathcal{R})\leq\eta l^{n-1}\log l \tag{5.11}$$

with η arbitrarily small. We have

$$\begin{aligned}\mathcal{J}(H,A_{l/2}\setminus\mathcal{R})\leq&4\int_{A_{l/2}\setminus\mathcal{R}}|\nabla\varphi|^2(V-U)^2+|\nabla(V-U)|^2dxdy\\&+\int_D W(u)+W(v)+C(v-u)^2dx\end{aligned} \tag{5.12}$$

with $D:=\mathcal{C}(\frac{l}{2},\frac{l}{2})\setminus\mathcal{C}(\frac{l}{2}-1,\frac{l}{2}-1)$. The second integral is bounded by Cl^{n-1}.

Next we bound the first integral. As in (5.9), u and v are sufficiently close to ±1 in $\mathcal{C}(\frac{l}{2},\frac{l}{2})$ away from a thin strip around $x_n=0$,

$$|v-u|\leq\gamma^2\quad\text{in }\ \mathcal{C}(l/2,l/2)\quad\text{if }\ |x_n|\geq C(\gamma)+\theta$$

with $C(\gamma)$ large, depending on the universal constants and γ. Then in the region

$$S:=\{1\leq y\leq\gamma^2(|x_n|-C(\gamma)-\theta)\},$$

the extensions U and V satisfy

$$|V-U|\leq C\gamma^2,\quad|\nabla(V-U)|\leq C\gamma^2y^{-1}. \tag{5.13}$$

At all other points we use that $|U|,|V|\leq1$, $|\nabla U|,|\nabla V|\leq C/(1+y)$ and we see from (5.12) that

$$\begin{aligned}\mathcal{J}(H,A_{l/2}\setminus\mathcal{R})&\leq Cl^{n-1}+C\int\gamma^2y^{-2}\chi_{(A_{l/2}\setminus\mathcal{R})\cap S}+(1+y)^{-2}\chi_{(A_{l/2}\setminus\mathcal{R})\setminus S}\,dxdy\\&\leq C(\gamma)l^{n-1}+C\gamma^2l^{n-1}\log l\leq\eta l^{n-1}\log l\end{aligned}$$

for all l large, provided that γ is chosen small, and (5.11) is proved. □

We conclude this section with the proof of the Lemma 5.1.

Proof of Lemma 5.1. The proof of (5.4) follows by the same argument of Lemma 5.2 above restricted to the case $n=1$ (now we denote x_n by t). First we may assume that V is minimizing the energy among functions which have prescribed boundary data on $\partial Q_{l+1}\setminus\{y=0\}$ and are constrained to (5.3) on $y=0$. We interpolate between V and G

$$H=(1-\varphi)G+\varphi V,$$

with φ defined as above and obtain

$$\mathcal{J}(H,Q_l)=\mathcal{J}(V,\mathcal{R})+\mathcal{J}(H,Q_l\setminus\mathcal{R}).$$

As in (5.12) we can use that in the region

$$S:=\{1\leq y\leq \gamma^2(|t|-l/2)\}$$

the functions V and G satisfy the estimate (5.13) and obtain

$$\mathcal{J}(H,Q_l)\leq C(\gamma)+C\gamma^2\log l\leq \frac{\gamma}{2}\log l.$$

Thus,

$$\mathcal{J}(V,\mathcal{R})\geq \mathcal{J}(H,Q_l)-\frac{\gamma}{2}\log l\geq \mathcal{J}(G,Q_l)-\frac{\gamma}{2}\log l,$$

and (5.4) follows by (5.2). Above we used that $H=G$ on $\partial Q_l\setminus\{y=0\}$ and the fact that G is a minimizer of $\mathcal{J}$ in Q_l.

For the second part we use $\bar{V}$, the monotone increasing rearrangement in the t direction of V. Denote by

$$\Gamma(D):=\{z=V(t,y)|(t,y)\in D\}\subset\mathbb{R}^3$$

the graph of V over the set D, and let T be the angular region

$$T:=\{y\geq|t-x_1|\}\cap B_{l^\sigma/2}.$$

Notice that our hypotheses imply that $|s_1-s_2|\geq l^\sigma$ provided that $c(\theta_0)$ is sufficiently small. This means that that the projection of $\Gamma(T)$ along the t direction is included in the projection of $\Gamma(Q_l\setminus T)$.

From the theory of monotone increasing rearrangements (see [7]) we obtain that

$$\mathcal{J}(V,Q_l)\geq \mathcal{J}(\bar{V},Q_l)+\int_T V_t^2\,dtdy.$$

On each horizontal segment ℓ_y of T at height $y\in[1,l^\sigma/4]$, we use that $V(t,y)$ and $G(t-x_1,y)$ are sufficiently close and obtain

$$\int_{\ell_y} V_t^2\,dt\geq \frac{c}{y},$$

hence

$$\int_T V_t^2\,dtdy\geq 2c_0(\sigma)\log l.$$

Notice that the rearrangement $\bar{V}$ still satisfies the hypothesis in part a), and then the conclusion follows from a).

6 Improvement of flatness

We state the improvement of flatness property of minimizers.

Theorem 6.1 (Improvement of flatness). *Let U be a minimizer of $\mathcal{J}$ and assume*

$$0\in\{u=0\}\cap\mathcal{C}(l,l)\subset\mathcal{C}(l,\theta), \tag{6.1}$$

and

$$\text{the balls of radius } C'l^2\theta^{-1} \ (C' \text{ universal}), \tag{6.2}$$

which are tangent to $\mathcal{C}(l,\theta)$ by below and above at $\pm\theta e_n$ are included in $\{u<0\}$ respectively $\{u>0\}$.

There exists universal integers $m_0\geq 1$, $m_1\geq 0$ such that if the balls of radius

$$R_m:=l_m^2\theta_m^{-1},\quad l_m:=2^m l,\quad \theta_m:=2^{\frac{3}{2}m}\theta,\quad m:=0,1,\cdots,m_1,$$

tangent to $\mathcal{C}(l_m,\theta_m)$ by below and above at $\pm\theta_m e_n$ are included in $\{u<0\}$ respectively $\{u>0\}$, and if

$$\theta l^{-1}=:\varepsilon\leq\varepsilon_1(\theta_0),\quad \theta\geq\theta_0,$$

with $\varepsilon_1(\theta_0)$ sufficiently small, then (6.1), (6.2) *hold for $\bar{l}$, $\bar{\theta}$ after a rotation with*

$$\{u=0\}\cap\mathcal{C}_\xi(\bar{l},\bar{l})\subset C_\xi(\bar{l},\bar{\theta}),\quad \bar{l}:=2^{-m_0}l,\quad \bar{\theta}:=2^{-\frac{3}{2}m_0}\theta.$$

Here $\xi\in\mathbb{R}^n$ is a unit vector and $C_\xi(\bar{l},\bar{\theta})$ represents the cylinder with axis ξ, base $\bar{l}$ and height $\bar{\theta}$.

As a consequence of this flatness theorem we obtain our main theorem.

Theorem 6.2. *Let U be a global minimizer of $\mathcal{J}$. Suppose that the* 0 *level set $\{u=0\}$ is asymptotically flat at ∞. Then the* 0 *level set is a hyperplane and u is one-dimensional.*

Proof. Without loss of generality assume $u(0)=0$. Fix $\theta_0>0$, and $\varepsilon\ll\varepsilon_1(\theta_0)$. From the hypotheses we can find l, θ large such that $\theta l^{-1}=\varepsilon$ and, after eventually a rotation, conditions (6.1), (6.2) hold for all l_m, θ_m

$$l_m:=2^m l,\quad \theta_m:=2^{\frac{3}{2}m}\theta,\quad \text{with } m\in\{m_1,m_1-1,\cdots,1-m_0\}.$$

Then, by Theorem 6.1, (6.1), (6.2) hold also for $m=-m_0$ after a rotation. It is easy to check that we can apply Theorem 6.1 repeatedly till the height of the cylinder becomes less than θ_0. We conclude that $\{u=0\}$ is trapped in a cylinder with flatness less than ε and height between $2^{-\frac{3}{2}}\theta_0$ and θ_0. We let first $\varepsilon\to 0$ and then $\theta_0\to 0$ and obtain the desired conclusion. □

Proof of Theorem 6.1. The proof is by compactness and it follows from Proposition 5.1 and Proposition 4.1. Assume by contradiction that there exist U_k, θ_k, l_k such that

a) U_k is a minimizer of $\mathcal{J}$, and satisfies (6.1), (6.2) for l_k, θ_k, together with the second hypothesis for $l_{k,m}$, $\theta_{k,m}$ and $m\in\{0,1,\cdots,m_{1,k}\}$,

b) $\theta_k\geq\theta_0$, $\theta_k l_k^{-1}=\varepsilon_k\to 0$, $m_{1,k}\to\infty$, as $k\to\infty$,

c) the conclusion of Theorem 6.1 does not hold for u_k with a constant m_0 depending only on n and C' which we will specify later.

Let A_k be the rescaling of the 0 level sets given by

$$\begin{aligned}&(x',x_n)\in\{u_k=0\}\mapsto(z',z_n)\in A_k,\\ &z'=x'l_k^{-1},\quad z_n=x_n\theta_k^{-1}.\end{aligned}$$

Claim 1: A_k has a subsequence that converges uniformly on $|z'|\leq 1/2$ to a set $A_\infty=\{(z',w(z')),\ |z'|\leq 1/2\}$ where w is a Holder continuous function.

Proof. Fix z_0', $|z_0'|\leq 1/2$. We apply Proposition 5.1 for the function u_k in the cylinder of base $B'_{l/2}(l_k z_0')$ and height $2\theta_k$ in which the set $\{u_k=0\}$ is trapped. Thus, there exist an increasing function $\varepsilon_0(\theta)>0$, $\varepsilon_0(\theta)\to 0$ as $\theta\to 0$, such that $\{u_k=0\}$ is trapped in the cylinder of base $B'_{l/2}(l_k z_0')$ and height $2(1-\frac{\omega_0}{2})\theta_k$ provided that $4\theta_k l_k^{-1}\leq\varepsilon_0(2\theta_k)$. Rescaling back we find that the oscillation of A_k in the z_n variable in $B'_{1/8}(z_0')$ is bounded by $2(1-\frac{\omega_0}{2})$. It is not difficult to see that we can apply the Harnack inequality repeatedly and we find that the oscillation of A_k in the z_n variable in $B'_{2^{-2m-1}}(z_0')$ is bounded by $2(1-\frac{\omega_0}{2})^m$ provided that

$$\varepsilon_k\leq 4^{-m-1}\varepsilon_0\left(2\left(1-\frac{\omega_0}{2}\right)^m\theta_k\right).$$

Since these inequalities are satisfied for all k large, the claim follows from a version of Arzela-Ascoli Theorem. ☐

Claim 2. The function w is harmonic (in the viscosity sense).

Proof. The proof is by contradiction. Fix a quadratic polynomial

$$z_n=P(z')=\frac{1}{2}z'^T Mz'+\xi\cdot z',\quad \|M\|<\delta^{-1},\quad 2|\xi|<\delta^{-1},$$

such that $trM>\delta$, $P(z')+\delta|z'|^2$ touches the graph of w, say, at 0 for simplicity, and stays below w in $|z'|<\delta$, for some small δ.

Thus, for all k large we find points $({z_k}',z_{kn})$ close to 0 such that $P(z')+const$ touches A_k by below at $({z_k}',z_{kn})$ and stays below it in $|z'-{z_k}'|<\delta/2$.

This implies that, after eventually a translation, there exists a surface

$$\Gamma:=\left\{x_n=\frac{\theta_k}{l_k^2}\sum\frac{a_i}{2}x_i^2+\frac{\theta_k}{l_k}\xi_k\cdot x'\right\},\quad |\xi_k|\leq\delta^{-1},\quad |a_i|\leq\delta^{-1},$$

with

$$\sum a_i>\delta$$

that touches $\{u_k=0\}$ at the origin and stays below it in the cylinder $\mathcal{C}(\frac{1}{2}\delta l_k,\theta_k)$.

Now we apply Proposition 4.1 with

$$\bar{l}:=\delta^3 l_k,\quad \bar{\theta}:=\theta_k,\quad \bar{R}:=\bar{l}^2\bar{\theta}^{-1},\quad \bar{\delta}:=\delta^2,\quad \bar{a}_i:=\frac{\theta_k}{l_k^2}a_i=\frac{\bar{\delta}^3}{\bar{R}}a_i,\quad \bar{b}'=\frac{\theta_k}{l_k}\xi_k.$$

The hypotheses are satisfied since

$$\bar{l}\le c(\delta)\varepsilon_k\bar{R}\le\bar{\delta}^3\bar{R},\quad \bar{l}\ge c(\theta_0)\bar{R}^{1/2}\ge\bar{R}^{1/3},$$

and

$$|\bar{a}_i|,|b'|\bar{l}^{-1}\le\delta^2\bar{R}^{-1}=\bar{\delta}\bar{R}^{-1}.$$

Moreover, by property a) above the balls of radius

$$\bar{R}_m=\bar{l}_m^2\bar{\theta}_m^{-1},\quad \text{with }\ \bar{l}_m:=2^m\bar{l},\quad \bar{\theta}_m:=2^{\frac{3}{2}m}\theta,\quad m\in\{0,1,\cdots,m_{k,1}\},$$

and tangent to $\mathcal{C}(\bar{l}_m,\bar{\theta}_m)$ are included in $\{u<0\}$ and respectively $\{u>0\}$. By Proposition 4.1 we conclude that

$$\sum\bar{a}_i\le\bar{\delta}^4\bar{R}^{-1}\Longrightarrow\sum a_i\le\bar{\delta}=\delta^2,$$

and we reached a contradiction, and Claim 2 is proved. □

Since w is harmonic, and $w(0)=0$,

$$|w-\xi'\cdot z'|\le C(n)|z'|^2,\quad |\xi'|\le C(n),$$

with $C(n)$ a constant depending only on n. We choose $2^{-m_0}=\eta$ sufficiently small such that

$$C(n)\eta^2\le\frac{1}{2}\eta^{3/2},\quad 4C(n)C'\eta^{\frac{1}{2}}\le 1.$$

Now it is easy to check that after rescaling back, and using the fact that A_k converge uniformly to the graph of w, the sets $\{u_k=0\}$ satisfy the conclusion of the Theorem 6.1 for all k large enough, and we reached a contradiction. □

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References

[1] X. Cabre and E. Cinti, Sharp energy estimates for nonlinear fractional diffusion equations, Calc. Var. Partial Differential Equations, 49 (2014), 233–269.

[2] E. De Giorgi, Convergence problems for functional and operators, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, (Rome, 1978), 131–188.

[3] D. De Silva and O. Savin, Quasi-Harnack Inequality, Preprint, arxiv 1803.10183.

[4] M. Del Pino, M. Kowalczyk and J. Wei, On De Giorgi Conjecture in dimension $N \geq 9$, Ann. Math., (2) (2011), 1485–1569.

[5] S. Dipierro, J. Serra and E. Valdinoci, Improvement of flatness for nonlocal phase transitions, Preprint arXiv:1611.10105.

[6] A. Figalli and J. Serra, On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4+1, Preprint, arXiv:1705.02781.

[7] B. Kawohl, Rearrangements and convexity of level sets in PDE, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985. iv+136 pp.

[8] G. Palatucci, O. Savin and E. Valdinoci, Local and global minimizers for a variational energy involving a fractional norm, Ann. Mat. Pura Appl., (4) 192 (2013), 673–718.

[9] L. Modica, Γ-convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 223–244, Pitagora, Bologna, 1979.

[10] O. Savin, Regularity of flat level sets in phase transitions, Ann. Math., (2) (2009), 41–78.

[11] O. Savin, Some remarks on the classification of global solutions with asymptotically flat level sets, Calc. Var. Partial Differential Equations, 56 (2017), Art. 141, 21 pages.

[12] O. Savin, Rigidity of minimizers in nonlocal phase transitions, Analysis and PDE, (to appear) arXiv:1610.09295.

[13] O. Savin and E. Valdinoci, Γ-convergence for nonlocal phase transitions, Ann. Inst. H. Poincare Anal. Non Linéaire, 29 (2012), 479–500.