Rigidity of Minimizers in Nonlocal Phase Transitions II

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Abstract. In this paper we extend the results of [12] to the borderline case $s = \frac{1}{2}$. We obtain the classification of global bounded solutions with asymptotically flat level sets for semilinear nonlocal equations of the type

$$\Delta^s u = W'(u) \quad \text{in} \quad \mathbb{R}^n,$$

where $W$ is a double well potential.

Key Words: De Giorgi’s conjecture, fractional Laplacian.

AMS Subject Classifications: 35J61

1 Introduction

We continue the study initiated in [12] for the classification of global bounded solutions with asymptotically flat level sets for nonlocal semilinear equations of the type

$$\Delta^s u = W'(u) \quad \text{in} \quad \mathbb{R}^n,$$

where $W$ is a double well potential.

The case $s \in (\frac{1}{2}, 1)$ was treated in [12] while $s \in (0, \frac{1}{2})$ was considered by Dipierro, Serra and Valdinoci in [5]. In this paper we obtain the classification of global minimizers with asymptotically flat level sets in the remaining borderline case $s = \frac{1}{2}$. All these works were motivated by the study of semilinear equations for the case of the classical Laplacian $s = 1$, and their connection with the theory of minimal surfaces, see [2, 4, 9, 10]. It turns out that when $s \in [\frac{1}{2}, 1)$, the rescaled level sets of $u$ still converge to a minimal surface while for $s \in (0, \frac{1}{2})$ they converge to an $s$-nonlocal minimal surface, see [13].

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We consider the Ginzburg-Landau energy functional with nonlocal interactions corresponding to $\Delta^{1/2}$,

$$J(u, \omega) = \frac{1}{4} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\overline{\omega} \times \overline{\omega})} \frac{(u(x) - u(y))^2}{|x-y|^{n+1}} dx dy + \int_{\omega} W(u) dx,$$

with $|u| \leq 1$, and $W$ a double-well potential with minima at 1 and $-1$ satisfying

$$W \in C^2([1,1]), \quad W(-1) = W(1) = 0, \quad W > 0 \text{ on } (-1,1),$$
$$W'(-1) = W'(1) = 0, \quad W''(-1) > 0, \quad W''(1) > 0.$$

Critical functions for the energy $J$ satisfy the Euler-Lagrange equation

$$\Delta^{1/2} u = W'(u),$$

where $\Delta^{1/2} u$ is defined as

$$\Delta^{1/2} u(x) = PV \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+1}} dy.$$

Our main result provides the classification of minimizers with asymptotically flat level sets.

**Theorem 1.1.** Let $u$ be a global minimizer of $J$ in $\mathbb{R}^n$. If the 0 level set $\{u = 0\}$ is asymptotically flat at $\infty$, then $u$ is one-dimensional.

The hypothesis that $\{u = 0\}$ is asymptotically flat means that there exist sequences of positive numbers $\theta_k, l_k$ and unit vectors $\xi_k$ with $l_k \to \infty$, $\theta_k l_k^{-1} \to 0$ such that

$$\{u = 0\} \cap B_{l_k} \subset \{|x \cdot \xi_k| < \theta_k\}.$$

By saying that $u$ is one-dimensional we understand that $u$ depends only on one direction $\xi$, i.e., $u = g(x \cdot \xi)$.

As in [12], we obtain several corollaries. We state two of them.

**Theorem 1.2.** A global minimizer of $J$ is one-dimensional in dimension $n \leq 7$.

**Theorem 1.3.** Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta^{1/2} u = W'(u),$$

such that

$$|u| \leq 1, \quad \partial_n u > 0, \quad \lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1.$$

Then $u$ is one-dimensional if $n \leq 8$. 