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Lower Bounds of Dirichlet Eigenvalues for General Grushin Type Bi-Subelliptic Operators

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Abstract. Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $X=(X_1,X_2,\cdots,X_m)$ be a system of general Grushin type vector fields defined on Ω and the boundary $\partial\Omega$ is non-characteristic for X. For $\Delta_X=\sum_{j=1}^m X_j^2$, we denote λ_k as the k-th eigenvalue for the bi-subelliptic operator Δ_X^2 on Ω. In this paper, by using the sharp sub-elliptic estimates and maximally hypoelliptic estimates, we give the optimal lower bound estimates of λ_k for the operator Δ_X^2.

Key Words: Eigenvalues, degenerate elliptic operators, sub-elliptic estimate, maximally hypoelliptic estimate, bi-subelliptic operator.

AMS Subject Classifications: 35J30, 35J70, 35P15

1 Introduction and main results

Let $X=(X_1,X_2,\cdots,X_m)$ be the system of general Grushin type vector fields, which is defined on an open domain W in $\mathbb{R}^n$ ($n\geq 2$).

Let $J=(j_1,\cdots,j_k)$, $1\leq j_i\leq m$ be a multi-index, $X^J=X_{j_1}X_{j_2}\cdots X_{j_k}$, we denote $|J|=k$ be the length of J, if $|J|=0$, then $X^J=id$. We introduce following function space (cf. [18,21,23]):

$$H_X^2(W)=\{u\in L^2(W)|X^Ju\in L^2(W),\ |J|\leq 2\}.$$

It is well known that $H_X^2(W)$ is a Hilbert space with norm $\|u\|^2_{H_X^2(W)}=\sum_{|J|\leq 2}\|X^Ju\|^2_{L^2(W)}$.

Assume the vector fields $X=(X_1,X_2,\cdots,X_m)$ satisfy Hörmander's condition :

Definition 1.1 (cf. [2,12])**.** We say that $X=(X_1,X_2,\cdots,X_m)$ satisfies the Hörmander's condition in W if there exists a positive integer Q, such that for any $|J|=k\leq Q$, X together with all k-th repeated commutators

$$X_J=[X_{j_1},[X_{j_2},[X_{j_3},\cdots,[X_{j_{k-1}},X_{j_k}]\cdots]]]$$

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span the tangent space at each point of W. Here Q is called the Hörmander index of X in W, which is defined as the smallest positive integer for the Hörmander's condition to be satisfied.

For any bounded open subset $\Omega\subset\subset W$, we define the subspace $H^2_{X,0}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^2_X(W)$. Since $\partial\Omega$ is smooth and non characteristic for X, we know that $H^2_{X,0}(\Omega)$ is well defined and also a Hilbert space. In this case, we also say that X satisfies the Hörmander's condition on Ω with Hörmander index $1\le Q<+\infty$. Thus X is a finitely degenerate system of vector fields on Ω and the finitely degenerate elliptic operator $\Delta_X=\sum_{i=1}^m X_i^2$ is a sub-elliptic operator.

The degenerate elliptic operator Δ_X has been studied by many authors, e.g., Hörmander [11], Jerison and Sánchez-Calle [13], Métivier [17], Xu [23]. More results for degenerate elliptic operators can be found in [2–6] and [9,10,12,14].

In this paper, we study the following eigenvalues problem for bi-subelliptic operators in $H^2_{X,0}(\Omega)$:

$$\begin{cases}\Delta_X^2 u=\lambda u & \text{in } \Omega,\\ u=0,\ Xu=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where X will be the following general Grushin type vector fields (see (1.5) and (1.7) below). In this case we know that for each j, X_j is formally skew-adjoint, i.e., $X_j^*=-X_j$. Then there exists a sequence of discrete eigenvalues $\{\lambda_j\}_{j\ge1}$ for the problem (1.1), which satisfying $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_k\cdots$ and $\lambda_k\to+\infty$ as $k\to+\infty$ (see Proposition 2.5 below).

In the classical case, if $X=(\partial_{x_1},\cdots,\partial_{x_n})$, then $\Delta_X^2=\Delta^2$ is the standard bi-harmonic operator. In this case our problem is motivated from the following classical clamped plate problem, namely

$$\begin{cases}\Delta^2 u=\lambda u & \text{in } \Omega,\\ u=\dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where $\Delta=\partial^2_{x_1}+\partial^2_{x_2}+\cdots+\partial^2_{x_n}$, $\frac{\partial u}{\partial\nu}$ denotes the derivative of u with respect to the outer unit normal vector ν on $\partial\Omega$.

For the eigenvalues of the clamped plate problem (1.2), Agmon [1] and Pleijel [20] showed the following asymptotic formula

$$\lambda_k\sim\frac{16\pi^4}{(B_n vol(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} \quad \text{as} \quad k\to+\infty, \tag{1.3}$$

where B_n denotes the volume of the unit ball in R^n. In 1985, Levine and Protter [15] proved that

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i\ge\frac{n}{n+4}\frac{16\pi^4}{(B_n vol(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}. \tag{1.4}$$