Some Notes on $k$-minimality

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Abstract. The concept of minimality is generalized in different ways, one of which is the definition of $k$-minimality. In this paper $k$-minimality is studied for minimal hypersurfaces of a Euclidean space under different conditions on the number of principal curvatures. We will also give a counterexample to $L_k$-conjecture.

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1 Introduction

Let $x : M \to \mathbb{E}^m$ be an isometric immersion from a Riemannian $n$-manifold into a Euclidean space. Denote the Laplacian, the position vector and the mean curvature vector field of $M$, respectively, by $\Delta, x$ and $\vec{H}$. Then, $M$ is called a biharmonic submanifold if $\Delta \vec{H} = 0$. Beltrami’s formula, $\Delta x = -n \vec{H}$, implies that every minimal submanifold of $\mathbb{E}^m$ is a biharmonic submanifold.

Chen initiated the study of biharmonic submanifolds in the mid 1980s [4]. Then, Chen and other authors proved that, in specific cases, a biharmonic submanifold is a minimal submanifold [4, 5, 7] and Chen introduced his famous conjecture [3]. This conjecture remains open, although the study thereof is active nowadays. Among other results, it is proved in [6] that Chen’s Conjecture is true for biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^m$. Furthermore, under a generic condition, Koiso and Urakawa [8] gave affirmative answer to Chen conjecture.

The linearized operator of $(k+1)$-th mean curvature of a hypersurface, i.e. $H_{k+1}$, is the $L_k$ operator. The $L_k$ operator is a natural generalization of Laplace operator for $k=1, \ldots, n$ [9, 10]. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from a connected orientable Riemannian hypersurface into the Euclidean space $\mathbb{E}^{n+1}$. It is proved that [1]

$$L_k x = (k+1) \binom{n}{k+1} H_{k+1} N,$$

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where $N$ is the unit normal vector field and $k=0,\ldots,n-1$. The $L_k$-conjecture is as follows.

$L_k$-Conjecture. Every $L_k$-biharmonic hypersurface, namely a Euclidean hypersurface $x: M^n \to \mathbb{E}^{n+1}$ satisfying the condition $L_k^2 x = 0$ for some $k=0,\ldots,n-1$, has zero $(k+1)$-th mean curvature.

A manifold with zero $(k+1)$-th mean curvature is called $k$-minimal for $k=0,\ldots,n-1$. In 2015, Aminian and Kashani [2] proved the $L_k$-conjecture for Euclidean hypersurfaces with at most two principal curvatures. They also proved the $L_k$-conjecture for $L_k$-finite type hypersurfaces.

In this paper, we prove that the $L_1$-conjecture is not true for a connected minimal hypersurface of a Euclidean space with arbitrary number of principal curvatures.

2 Preliminaries

In this section, we recall some standard definitions and results from Riemannian geometry. Let $n \geq 2$ and suppose $x: M^n \to \mathbb{E}^{n+1}$ is an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$.

Let $A$ be the shape operator of this immersion and $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of this self-adjoint operator. The mean curvature of $M$ is given by

$$nH = \text{trace } A = \lambda_1. \ldots \lambda_n.$$ 

The $k$-th mean curvature of $M$ is also defined by

$$\binom{n}{k} H_k = s_k,$$

where $s_0 = 1$ and $s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}$ for $k = 1,\ldots,n$. It is obvious that $H_1 = H$ and $S = n(n-1)H_2$, where $S$ is the scalar curvature of $M$.

The Newton transformations $P_k: C^\infty(TM^n) \to C^\infty(TM^n)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - A \circ P_{k-1}, \quad 1 \leq k \leq n.$$ 

Therefore,

$$P_k = \sum_{i=0}^{k} (-1)^i s_{k-i} A^i, \quad 1 \leq k \leq n.$$ 

Thus the Cayley-Hamilton theorem implies that $P_n = 0$. It is well known that each $P_k$ is a self-adjoint linear operator which commutes with $A$. For $k = 0, \ldots, n$, the second
order linear differential operator $L_k: C^\infty(M^n) \to C^\infty(M^n)$, as a natural generalization of the Laplace operator on Euclidean hypersurface $M$, is defined by

$$L_k f = \text{trace}(P_k \circ \nabla^2 f),$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of $f$ and is defined by

$$\langle (\nabla^2 f)X, Y \rangle = \langle \nabla X(\nabla f), Y \rangle$$

for all vector fields $X, Y \in C^\infty(TM^n)$. Here $\nabla f$ is the gradient vector field of $f$.

When $k = 0$, $L_0 = \Delta$. In this case, we have also $A \circ P_0 = A$ and $L_0^2 x = 0$, which means, $M^n$ is a biharmonic hypersurface.

### 3 1-minimality and counterexample

In this section, we first consider the minimal hypersurfaces of $\mathbb{E}^n$ with three distinct principal curvatures.

**Theorem 3.1.** Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$. If $M^n$ is a minimal hypersurface with three distinct principal curvatures, then $M^n$ cannot be 1-minimal.

**Proof.** Let $\lambda_1 = ... = \lambda_p = \alpha$, $\lambda_{p+1} = ... = \lambda_{p+q} = \beta$, $\lambda_{p+q+1} = ... = \lambda_n = \gamma$ be principal curvatures of $M^n$, for $1 \leq p \leq n-2$ and $1 \leq q \leq n-2$. Then we have

$$s_2 = \frac{1}{2} p(p-1) \alpha^2 + \frac{1}{2} q(q-1) \beta^2 + \frac{1}{2} (n-(p+q))(n-(p+q)-1) \gamma^2 + pq \alpha \beta$$

$$+ (n-(p+q))(p \alpha + q \beta) \gamma.$$ 

The minimality of $M^n$ yields

$$\gamma = -\frac{(p \alpha + q \beta)}{n - (p + q)}.$$ 

So we have

$$s_2 = -\frac{1}{2(n-(p+q))}(p \alpha + q \beta)^2 - \frac{1}{2}(p \alpha^2 + q \beta^2). \quad (3.1)$$

Now if $M^n$ is 1-minimal, then (3.1) implies that

$$(p \alpha + q \beta)^2 = -(n-(p+q))(p \alpha^2 + q \beta^2).$$

This concludes that $\alpha = \beta = \gamma = 0$, which contradicts the assumption. \(\square\)

Theorem 3.1 can be generalized to an arbitrary number of principal curvatures as in the following theorem,
Theorem 3.2. Let \( x : M^n \rightarrow \mathbb{E}^{n+1} \) be an isometric immersion from an \( n \)-dimensional connected Riemannian manifold \( M^n \) into Euclidean space \( \mathbb{E}^{n+1} \). If \( M^n \) is a minimal hypersurface with \( k \), \( k > 1 \), distinct principal curvatures, then \( M^n \) cannot be 1-minimal.

Proof. Let \( \lambda_1 = \ldots = \lambda_{p_1} = \alpha_1, \lambda_{p_1+1} = \ldots = \lambda_{p_1+p_2} = \alpha_2, \ldots, \lambda_{\Sigma_{i=1}^{k-1} p_i + 1} = \ldots = \lambda_n = \alpha_k \) be principal curvatures of \( M^n \), for \( 1 \leq p_i \leq n - (k - 1), 1 < i < k - 1 \), and let \( p_k = n - \Sigma_{i=1}^{k-1} p_i \). Therefor we have

\[
 s_2 = \frac{1}{2} \sum_{i=1}^{k-1} p_i (p_i - 1) \alpha_i^2 + \frac{1}{2} p_k (p_k - 1) \alpha_k^2 + \sum_{1 \leq i < j \leq k-1} p_i p_j \alpha_i \alpha_j + p_k (\sum_{i=1}^{k-1} p_i \alpha_i) \alpha_k.
\]

We obtain from minimality of \( M^n \),

\[
 \alpha_k = -\sum_{i=1}^{k-1} \frac{p_i \alpha_i}{p_k},
\]

so we have

\[
 s_2 = -\frac{1}{2p_k} \left( \sum_{i=1}^{k-1} p_i \alpha_i \right)^2 - \frac{1}{2} \left( \sum_{i=1}^{k-1} p_i \alpha_i^2 \right). \tag{3.2}
\]

Now if \( M^n \) is 1-minimal, then (3.2) implies that

\[
 (\sum_{i=1}^{k-1} p_i \alpha_i)^2 = -p_k \sum_{i=1}^{k-1} p_i \alpha_i^2.
\]

This concludes that \( \alpha_i = 0 \), for \( i = 1, \ldots, k \), which contradicts the assumption. \( \square \)

An immediate corollary for this section is stated as follows.

**Corollary 3.1.** Let \( x : M^n \rightarrow \mathbb{E}^{n+1} \) be an isometric immersion from an \( n \)-dimensional connected Riemannian manifold \( M^n \) into Euclidean space \( \mathbb{E}^{n+1} \). If \( M^n \) is a minimal hypersurface, then \( M^n \) is 1-minimal if and only if \( M^n \) has exactly one vanishing principal curvature.

We conclude this section with a Counterexample for \( L_k \)-Conjecture.

**Counterexample.** By Corollary 3.1, a connected minimal \( L_1 \)-biharmonic hypersurface of \( \mathbb{E}^{n+1} \) with at least one non zero principal curvature cannot be 1-minimal.

### 4 2-minimality

In this section we study the property of 2-minimality for some hypersurfaces of Euclidean spaces in the specific cases.
Theorem 4.1. Let $n$ be odd and $x: M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$. If $M^n$ is a minimal hypersurface with two distinct principal curvatures, then $M^n$ cannot be 2-minimal.

Proof. According to the calculation on the page 4 of [2], we have

$$s_3 = \sum_{i=0}^{n} \binom{m}{i} \left( \frac{n-m}{3-i} \right) \alpha^i \beta^{3-i}.$$  

From the minimality of $M^n$, it follows that $\alpha = -\frac{n-m}{m} \beta$. Therefore

$$s_3 = \sum_{i=0}^{n} \binom{m}{i} \left( \frac{n-m}{3-i} \right) (-1)^i \left( \frac{n-m}{m} \right)^i \beta^3. \quad (4.1)$$

Now from (4.1), if $M^n$ is 2-minimal, then $\alpha = \beta = 0$, which yields a contradiction. \qed

In Theorem 4.1, if $n$ is even, then we have another theorem.

Theorem 4.2. Let $n$ be even and $x: M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$. Assume also that $M^n$ is a minimal hypersurface with two distinct principal curvatures. If the multiplicity of principal curvatures are equal, then $M^n$ is 2-minimal. Otherwise, $M^n$ cannot be 2-minimal.

Proof. We have

$$\sum_{i=0}^{n} \binom{m}{i} \left( \frac{n-m}{3-i} \right) (-1)^i = 0,$$

thus if the multiplicity of two principal curvatures are equal, then $s_3 = 0$, by (4.1). This means that, independent of $\beta$, $M^n$ is 2-minimal. In other cases, the proof is similar to the proof of Theorem 4.1. \qed

For three distinct principal curvatures in a spacial case, we have the following theorem.

Theorem 4.3. Let $n \geq 5$ and $x: M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$. If $M^n$ is a minimal hypersurface with three distinct principal curvatures of multiplicity $\{ n-2, 1, 1 \}$, then $M^n$ is 2-minimal if and only if the principal curvature of multiplicity $n-2$ is vanish.

Proof. Let $\alpha, \beta, \gamma$ be principal curvatures of $M^n$ with multiplicity $n-2, 1, 1$ respectively. According to definition, we have

$$s_3 = \frac{(n-2)!}{3!(n-3)!} \alpha^3 + \frac{(n-2)(n-3)}{2} \alpha^2 (\beta + \gamma) + (n-2) \alpha \beta \gamma. \quad (4.2)$$
Since $M^n$ is minimal we have $\alpha = -\frac{\beta + \gamma}{n-2}$. So by substitution in (4.2) we get,

$$s_3 = T(\beta + \gamma)^3 - (\beta + \gamma)\beta\gamma = (\beta + \gamma)(T\beta^2 + T\gamma^2 + (2T-1)\beta\gamma),$$

(4.3)

where $T = \frac{n^2 - 4n + 3}{3(n-2)^2}$. Because of non-vanishing of the second parenthesis in (4.3) for $n \geq 5$, if $M^n$ is 2-minimal, we conclude that $\beta = -\gamma$ and $\alpha = 0$. The converse is obvious from (4.2).

A direct computation shows that the result of Theorem 4.3 is also true for $n = 3$ and $n = 4$.

The last theorem is about 2-minimality property for minimal hypersurfaces with three distinct principal curvatures, when the dimension of hypersurface is a multiple of three.

**Theorem 4.4.** Let $n = 3m$, for some $m \geq 3$ and $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an $n$-dimensional connected Riemannian manifold $M^n$ into Euclidean space $\mathbb{E}^{n+1}$. Assume also that $M^n$ is a minimal hypersurface with three principal curvatures of equal multiplicity. Then $M^n$ is 2-minimal if at least one of the principal curvatures is zero.

**Proof.** Let $\alpha, \beta, \gamma$ be principal curvatures of $M^n$. By definition of $s_3$, we have,

$$s_3 = \left( \begin{array}{c} m \\ 3 \end{array} \right) (\alpha^3 + \beta^3 + \gamma^3) + m \left( \begin{array}{c} m \\ 2 \end{array} \right) (\alpha^2 \beta + \alpha \beta^2 + \alpha^2 \gamma + \alpha \gamma^2 + \beta^2 \gamma + \beta \gamma^2) + m^3 \alpha \beta \gamma.$$

Because of minimality of $M^n$, we have also $\alpha = -(\beta + \gamma)$. Therefore we get

$$s_3 = -m(\beta + \gamma)\beta\gamma.$$

It is concluded that if $M^n$ is 2-minimal, then one of the principal curvatures is zero.

In Theorem 4.4, for the case $m = 2$, 2-minimality of $M^n$ is a direct result of its minimality.

**References**


