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Some Notes on k-minimality

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Abstract. The concept of minimality is generalized in different ways, one of which is the definition of k-minimality. In this paper k-minimality is studied for minimal hypersurfaces of a Euclidean space under different conditions on the number of principal curvatures. We will also give a counterexample to L_k-conjecture.

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1 Introduction

Let $x: M \to \mathbb{E}^m$ be an isometric immersion from a Riemannian $n-$manifold into a Euclidean space. Denote the Laplacian, the position vector and the mean curvature vector field of M, respectively, by Δ, x and $\vec{H}$. Then, M is called a biharmonic submanifold if $\Delta\vec{H}=0$. Beltrami's formula, $\Delta x = -n\vec{H}$, implies that every minimal submanifold of $\mathbb{E}^m$ is a biharmonic submanifold.

Chen initiated the study of biharmonic submanifolds in the mid 1980s [4]. Then, Chen and other authors proved that, in specific cases, a biharmonic submanifold is a minimal submanifold [4,5,7] and Chen introduced his famous conjecture [3]. This conjecture remains open, although the study thereof is active nowadays. Among other results, it is proved in [6] that Chen's Conjecture is true for biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^m$. Furthermore, under a generic condition, Koiso and Urakawa [8] gave affirmative answer to Chen conjecture.

The linearized operator of $(k+1)$-th mean curvature of a hypersurface, i.e. H_{k+1}, is the $\mathrm{L_k}$ operator. The L_k operator is a natural generalization of Laplace operator for $k=1,...,n$ [9,10]. Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from a connected orientable Riemannian hypersurface into the Euclidean space $\mathbb{E}^{n+1}$. It is proved that [1]

$$L_k x = (k+1)\begin{pmatrix} n \\ k+1 \end{pmatrix} H_{k+1} N,$$

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where N is the unit normal vector field and $k=0,..,n-1$.The L_k-conjecture is as follows.

L_k-Conjecture. Every L_k-biharmonic hypersurface, namely a Euclidean hypersurface $x: M^n \to \mathbb{E}^{n+1}$ satisfying the condition $L_k^2 x = 0$ for some $k=0,...,n-1$, has zero $(k+1)$-th mean curvature.

A manifold with zero $(k+1)$-th mean curvature is called k-minimal for $k=0,..,n-1$. In 2015, Aminian and Kashani [2] proved the L_k-conjecture for Euclidean hypersurfaces with at most two principal curvatures. They also proved the L_k-conjecture for L_k-finite type hypersurfaces.

In this paper, we prove that the L_1-conjecture is not true for a connected minimal hypersurface of a Eucldean space with arbitrary number of principal curvatures.

2 Preliminaries

In this section, we recall some standard definitions and results from Riemannian geometry. Let $n \geq 2$ and suppose $x: M^n \to \mathbb{E}^{n+1}$ is an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$.

Let A be the shape operator of this immersion and $\lambda_1,...,\lambda_n$ be the eigenvalues of this self-adjoint operator. The mean curvature of M is given by

$$nH = \text{trace } A = \lambda_1. \ ... \ .\lambda_n.$$

The $k-$th mean curvature of M is also defined by

$$\binom{n}{k} H_k = s_k,$$

where $s_0 = 1$ and $s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}. \ \dots \ .\lambda_{i_k}$, for $k=1,...,n$. It is obvious that $H_1 = H$ and $S = n(n-1)H_2$, where S is the scalar curvature of M.

The Newton transformations $P_k : C^\infty(TM^n) \to C^\infty(TM^n)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - A \circ P_{k-1}, \ \ 1 \leq k \leq n.$$

Therefore,

$$P_k = \sum_{i=0}^{k} (-1)^i s_{k-i} A^i, \ \ 1 \leq k \leq n.$$

Thus the Cayley-Hamilton theorem implies that $P_n = 0$. It is well known that each P_k is a self-adjoint linear operator which commutes with A. For $k=0$, ..., n, the second order linear differential operator $L_k : C^\infty(M^n) \to C^\infty(M^n)$, as a natural generalization of the Laplace operator on Euclidean hypersurface M, is defined by

$$L_k f = \text{trace}(P_k \circ \bigtriangledown^2 f),$$

where $\bigtriangledown^2 f$ is metrically equivalent to the Hessian of f and is defined by

$$\langle (\bigtriangledown^2 f)X, Y\rangle = \langle \bigtriangledown_X(\bigtriangledown f), Y\rangle$$

for all vector fields $X, Y \in C^\infty(TM^n)$. Here $\bigtriangledown f$ is the gradient vector field of f.

When $k=0$, $L_0 = \triangle$. In this case, we have also $A \circ P_0 = A$ and $L_0^2 x = 0$, which means, M^n is a biharmonic hypersurface.

3 1-minimality and counterexample

In this section, we first consider the minimal hypersurfaces of $\mathbb{E}^n$ with three distinct principal curvatures.

Theorem 3.1. *Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. If M^n is a minimal hypersurface with three distinct principal curvatures, then M^n cannot be 1-minimal.*

Proof. Let $\lambda_1 = ... = \lambda_p = \alpha$, $\lambda_{p+1} = ... = \lambda_{p+q} = \beta$, $\lambda_{p+q+1} = ... = \lambda_n = \gamma$ be principal curvatures of M^n, for $1 \le p \le n-2$ and $1 \le q \le n-2$. Therefor we have

$$\begin{aligned} s_2 = &\frac{1}{2}p(p-1)\alpha^2 + \frac{1}{2}q(q-1)\beta^2 + \frac{1}{2}(n-(p+q))(n-(p+q)-1)\gamma^2 + pq\alpha\beta \\ &+(n-(p+q))(p\alpha+q\beta)\gamma. \end{aligned}$$

The minimality of M^n yields

$$\gamma = -\frac{(p\alpha+q\beta)}{n-(p+q)}.$$

So we have

$$s_2 = -\frac{1}{2(n-(p+q))}(p\alpha+q\beta)^2 - \frac{1}{2}(p\alpha^2+q\beta^2). \tag{3.1}$$

Now if M^n is 1-minimal, then (3.1) implies that

$$(p\alpha+q\beta)^2 = -(n-(p+q))(p\alpha^2+q\beta^2).$$

This concludes that $\alpha = \beta = \gamma = 0$, which contradicts the assumption. □

Theorem 3.1 can be generalized to an arbitrary number of principal curvatures as in the following theorem,

Theorem 3.2. *Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. If M^n is a minimal hypersurface with k, $k>1$, distinct principal curvatures, then M^n cannot be 1-minimal.*

Proof. Let $\lambda_1 = ... = \lambda_{p_1} = \alpha_1$, $\lambda_{p_1+1} = ... = \lambda_{p_1+p_2} = \alpha_2, ..., \lambda_{\sum_{i=1}^{k-1} p_i+1} = ... = \lambda_n = \alpha_k$ be principal curvatures of M^n, for $1 \le p_i \le n-(k-1)$, $1 \le i < k-1$, and let $p_k = n - \Sigma_{i=1}^{k-1} p_i$. Therefor we have

$$s_2 = \frac{1}{2}\sum_{i=1}^{k-1} p_i(p_i-1)\alpha_i^2 + \frac{1}{2} p_k(p_k-1)\alpha_k^2 + \sum_{1\le i\le j\le k-1} p_i p_j \alpha_i \alpha_j + p_k(\sum_{i=1}^{k-1} p_i\alpha_i)\alpha_k.$$

We obtain from minimality of M^n,

$$\alpha_k = -\frac{\sum_{i=1}^{k-1} p_i\alpha_i}{p_k},$$

so we have

$$s_2 = \frac{-1}{2p_k}(\sum_{i=1}^{k-1} p_i\alpha_i)^2 - \frac{1}{2}(\sum_{i=1}^{k-1} p_i\alpha_i^2). \tag{3.2}$$

Now if M^n is 1-minimal, then (3.2) implies that

$$(\sum_{i=1}^{k-1} p_i\alpha_i)^2 = -p_k \sum_{i=1}^{k-1} p_i\alpha_i^2.$$

This concludes that $\alpha_i = 0$, for $i = 1,...,k$, which contradicts the assumption. □

An immediate corollary for this section is stated as follows.

Corollary 3.1. Let $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$.If M^n is a minimal hypersurface, then M^n is 1-minimal if and only if M^n has exactly one vanishing principal curvature.

We conclude this section with a Counterexample for L_k-Conjecture.

Counterexample. By Corollary 3.1, a connected minimal L_1-biharmonic hypersurface of $\mathbb{E}^{n+1}$ with at least one non zero principal curvature cannot be 1-minimal.

4 2-minimality

In this section we study the property of 2-minimality for some hypersurfaces of Euclidean spaces in the specific cases.

Theorem 4.1. *Let n be odd and $x\colon M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. If M^n is a minimal hypersurface with two distinct principal curvatures, then M^n cannot be 2-minimal.*

Proof. According to the calculation on the page 4 of [2], we have

$$s_3 = \sum_{i=0}^{n} \binom{m}{i} \binom{n-m}{3-i} \alpha^i \beta^{3-i}.$$

From the minimality of M^n, it follows that $\alpha = -\frac{n-m}{m}\beta$. Therefore

$$s_3 = \sum_{i=0}^{n} \binom{m}{i} \binom{n-m}{3-i} (-1)^i \left(\frac{n-m}{m}\right)^i \beta^3. \tag{4.1}$$

Now from (4.1), if M^n is 2-minimal, then $\alpha = \beta = 0$, which yields a contradiction. □

In Theorem 4.1, if n is even, then we have another theorem.

Theorem 4.2. *Let n be even and $x\colon M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. Assume also that M^n is a minimal hypersurface with two distinct principal curvatures. If the multiplicity of principal curvatures are equal, then M^n is 2-minimal. Otherwise, M^n cannot be 2-minimal.*

Proof. We have

$$\sum_{i=0}^{n} \binom{m}{i} \binom{m}{3-i} (-1)^i = 0,$$

thus if the multiplicity of two principal curvatures are equal, then $s_3 = 0$, by (4.1). This means that, independent of β, M^n is 2-minimal. In other cases, the proof is similar to the proof of Theorem 4.1. □

For three distinct principal curvatures in a spacial case, we have the following theorem.

Theorem 4.3. *Let $n \geq 5$ and $x\colon M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. If M^n is a minimal hypersurface with three distinct principal curvatures of multiplicity $\{n-2,1,1\}$, then M^n is 2-minimal if and only if the principal curvature of multiplicity $n-2$ is vanish.*

Proof. Let α,β,γ be principal curvatures of M^n with multiplicity $n-2,1,1$ respectively. According to definition, we have

$$s_3 = \frac{(n-2)!}{3!(n-5)!}\alpha^3 + \frac{(n-2)(n-3)}{2}\alpha^2(\beta+\gamma) + (n-2)\alpha\beta\gamma. \tag{4.2}$$

Since M^n is minimal we have $\alpha = -\dfrac{\beta+\gamma}{n-2}$. So by substitution in (4.2) we get,

$$s_3 = T(\beta+\gamma)^3 - (\beta+\gamma)\beta\gamma = (\beta+\gamma)(T\beta^2 + T\gamma^2 + (2T-1)\beta\gamma), \tag{4.3}$$

where $T = \frac{n^2-4n+3}{3(n-2)^2}$. Because of non-vanishing of the second parenthesis in (4.3) for $n \geq 5$, if M^n is 2-minimal, we conclude that $\beta = -\gamma$ and $\alpha = 0$. The converse is obvious from (4.2). □

A direct computation shows that the result of Theorem 4.3 is also true for $n=3$ and $n=4$.

The last theorem is about 2-minimality property for minimal hypersurfaces with three distinct principal curvatures, when the dimension of hypersurface is a multiple of three.

Theorem 4.4. *Let $n=3m$, for some $m \geq 3$ and $x: M^n \to \mathbb{E}^{n+1}$ be an isometric immersion from an n-dimensional connected Riemannian manifold M^n into Euclidean space $\mathbb{E}^{n+1}$. Assume also that M^n is a minimal hypersurface with three principal curvatures of equal multiplicity. Then M^n is 2-minimal if at least one of the principal curvatures is zero.*

Proof. Let α, β, γ be principal curvatures of M^n. By definition of s_3, we have,

$$s_3 = \binom{m}{3}(\alpha^3+\beta^3+\gamma^3) + m\binom{m}{2}(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2) + m^3\alpha\beta\gamma.$$

Because of minimality of M^n, we have also $\alpha = -(\beta+\gamma)$. Therefore we get

$$s_3 = -m(\beta+\gamma)\beta\gamma.$$

It is concluded that if M^n is 2-minimal, then one of the principal curvatures is zero. □

In Theorem 4.4, for the case $m=2$, 2-minimality of M^n is a direct result of its minimality.

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