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Regularity Criteria on the 2D Anisotropic Magnetic Bénard Equations

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Abstract. In this paper, we study the global regularity issue of two dimensional incompressible magnetic Bénard equations with partial dissipation and magnetic diffusion. It remains open whether the smooth initial data produce solutions that are globally regular in time for all values of the parameters involved in the equations. We present conditional global regularity of the solutions. Moreover, we prove the global regularity for the slightly regularized system.

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Key words: Global regularity, magnetic Bénard equations, partial dissipation

1 Introduction

This paper aims the global regularity of two dimensional magnetic Bénard equations. The standard two-dimensional incompressible magnetic Bénard equations can be written as

$$\begin{cases} u_t+u\cdot\nabla u=-\nabla p+\nu\Delta u+b\cdot\nabla b+\theta e_2, \\ b_t+u\cdot\nabla b=\eta\Delta b+b\cdot\nabla u, \\ \partial_t\theta+(u\cdot\nabla)\theta-\kappa\Delta\theta=u\cdot e_2, \\ \nabla\cdot u=0, \quad \nabla\cdot b=0, \\ u(x,y,0)=u_0(x,y),\ b(x,y,0)=b_0(x,y),\ \theta(x,y,0)=\theta_0(x,y), \end{cases} \tag{1.1}$$

where $(x,y)\in\mathbb{R}^2$, $t\geq 0$, $u=(u_1(x,y,t),u_2(x,y,t))$ denotes the 2D velocity field, $p=p(x,y,t)$ the pressure, $b=(b_1(x,y,t),b_2(x,y,t))$ the magnetic field, $\theta(x,y,t)$ the temperature, $e_2=(0,1)^T$ vertical unit vector, and ν, η and κ are nonnegative real parameters.

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A generalized 2D Magnetic Bénard equations can be written as

$$\begin{cases} u_t+u\cdot\nabla u=-\nabla p+\nu_1 u_{xx}+\nu_2 u_{yy}+b\cdot\nabla b+\theta e_2, \\ b_t+u\cdot\nabla b=\eta_1 b_{xx}+\eta_2 b_{yy}+b\cdot\nabla u, \\ \partial_t\theta+(u\cdot\nabla)\theta-\kappa_1\partial_{xx}\theta-\kappa_2\partial_{yy}\theta=u_2, \\ \nabla\cdot u=0, \quad \nabla\cdot b=0, \\ u(x,y,0)=u_0(x,y),\ b(x,y,0)=b_0(x,y),\ \theta(x,y,0)=\theta_0(x,y). \end{cases} \tag{1.2}$$

If $\nu_1=\nu_2=\nu$ and $\eta_1=\eta_2=\eta$, and $\kappa_1=\kappa_2=\kappa$, then (1.2) reduces to the standard magnetic Bénard equations (1.1). This generalization is capable of modeling the motion of anisotropic fluids for which the diffusion properties in different directions are different.

In the absence of θ, the magnetic Bénard equation reduces to magneto-hydrodynamic (MHD) equation. When all four parameters ν_1, ν_2, η_1, and η_2 are positive, the global regularity of the classical solution to 2D MHD equations has been established, see, e.g., [7], [19]. On the other hand, it remains a remarkable open problem whether classical solutions of the two-dimensional inviscid MHD equations, with all four parameters equal to zero, preserve their regularity for all time or have finite time blowup. Many attempts have been made but there are no any satisfactory results concerning the regularity of the solution. When $\nu_1>0$, $\nu_2=0$, $\eta_1=0$ and $\eta_2>0$ or when $\nu_1=0$, $\nu_2>0$, $\eta_1>0$ and $\eta_2=0$, the global regularity was established by Cao and Wu in [2]. Cao, Regmi, and Wu studied two dimensional MHD equations with horizontal dissipation and horizontal diffusion in [1]. They proved that any possible blow-up can be controlled by the L^∞-norm of the horizontal components.

There are numerous papers related to two dimensional MHD equations [1–8, 16, 20, 23, 25] and references therein, however only few papers are available related to magnetic Bénard equations. Y. Zhou et al. in [32] obtained the global regularity results related to the 2D magnetic Bénard problem with zero thermal conductivity. The authors used energy estimates as well as a well known property of Hardy space and Bounded Mean Value Oscillation (BMO) to prove the global regularity. Very recently, J. Cheng and L. Du in [6] proved the global well-posedness of the 2D Magnetic Bénard equations with mixed partial viscosity which included vertical or horizontal magnetic diffusion but no thermal diffusivity. The authors also obtained the global regularity as well as some conditional regularity of strong solutions of the problem with mixed partial viscosity, thus extending the existing result of the problem with the full dissipation. Likewise, the global regularity of generalized magnetic Bénard problem was studied by Y. Yamazaki in [28] by extending the existing results on Boussinesq equation and magneto-hydrodynamic equations. The author studied the problem with fractional Laplacian and logarithmic super criticality. The author showed that when the diffusive term has a full Laplacian, then a sufficiently smooth initial data evolves into a smooth solution under certain conditions. The author also presented additional global regularity criteria for the velocity field, magnetic field and the temperature field.