The Convexity of a Fully Nonlinear Operator and Its Related Eigenvalue Problem

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Abstract. We first get an existence and uniqueness result for a nonlinear eigenvalue problem. Then, we establish the constant rank theorem for the problem and use it to get a convexity property of the solution.

AMS subject classifications: 35J15, 35P30, 52A99

Key words: Hessian operator, eigenvalue problem, constant rank theorem, strict convexity.

1 Introduction

Let $\Omega \subset R^3$ be a strictly convex bounded domain in R^3 with smooth boundary. We consider the following eigenvalue problem

$$\begin{cases} \sigma_2(W_{ij}(D^2u)) = \lambda(-u)^2 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where $D^2u = (u_{ij})$ is the hessian matrix of u, $(W_{ij}(D^2u))$ is a symmetric matrix defined as

$$(W_{ij}) = \begin{bmatrix} u_{11}+u_{22} & u_{32} & -u_{31} \\ u_{23} & u_{11}+u_{33} & u_{21} \\ -u_{13} & u_{12} & u_{22}+u_{33} \end{bmatrix} \tag{1.2}$$

and σ_2 is the 2-nd hessian operator (i.e. $\sigma_2(S) =$ the sum of the 2-principal minors of S for any 3×3 symmetric matrix S). We first prove an existence and uniqueness result for (1.1). Then we get some convexity result for the solution of it.

The eigenvalue problem played an important role in partial differential equations and had been studied by many authors (see, e.g., [21, 24, 25, 33]). Lions [25] first got

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the existence and uniqueness result for the eigenvalue problem of Monge-Ampère equation. Later on, Wang [33] (and Geng-Yu-Qu [12]) generalized this result to the k-hessian equations. In this paper, we get similar results for (1.1). Another important problem in PDE is the convexity problem, which connects the geometric properties to geometric inequalities. One powerful tool to study the convexity is the constant rank theorem. Caffarelli-Friedman [6] proved a constant rank theorem for convex solutions of quasilinear elliptic equations in R^2. Meanwhile, a similar result was discovered by Yau [30]. Korevaar-Lewis [23] generalized their results to R^n. Later on, Caffarelli-Guan-Ma [8] and Bian-Guan [4] established the constant rank theorem for a class of fully nonlinear equations. Related to our problems, Liu-Ma-Xu [26] established the constant rank theorem for the eigenvalue problem related to $k-Hessian$ equations for $k=2$ in dimension $n=3$. In this paper, we would get a convexity result for (1.1) similar to [26].

Our another motivation to study (1.1) comes from the concept of $k-convex$ solutions introduced by Harvey-Lawson [18] who introduced some general convexity on the solutions of the nonlinear elliptic Dirichlet problem. In their definition, a C^2 function u is said to be $k-convex$ if the sum of any k eigenvalues of its hessian matrix is nonnegative. Recently, Han-Ma-Wu [16], Tosatti-Weinkove [31] studied a similar "convexity"-the $n-1$ plurisubharmonicity for C^2 functions defined on $\Omega \subset C^n$ (i.e. the sum of any $n-1$ eigenvalues of the complex hessian $(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j})$ is positive) and [31] used it to study the form-type Calabi-Yau equation (see [10, 11]). The $k-convexity$ is related to (1.1) in the sense that if we note the eigenvalues of $D^2 u$ by λ_i, $(i=1,2,3)$. Then, by an orthogonal transformation, it is easy to know that the three eigenvalues of $(W_{ij}(D^2 u))$ are $\lambda_1+\lambda_2, \lambda_1+\lambda_3, \lambda_2+\lambda_3$. So, u is $2-convex$ if and only if $(W_{ij}(D^2 u))$ is positive semi-definite. For our purpose here, we do not need $(W_{ij}(D^2 u))$ to be positive semi-definite. Instead, we only need $(W_{ij}(D^2 u)) \in \Gamma_2$ (the definition of Γ_2 will be given below) for the operator $F(D^2 u) = \sigma_2(W_{ij}(D^2 u))$ to be elliptic on u.

We see that the operator $F(D^2 u) = \sigma_2(W_{ij}(D^2 u))$ is a combination of the $2-hessian$ operator σ_2 and a linear one. In our proof of the theorems, we will use the elementary properties of the hessian operator repeatedly. So, let us state some preliminary knowledge that will be used below.

For $1 \leq k \leq n$, let σ_k be the $k-th$ elementary symmetric function, i.e.

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \qquad \forall \, \lambda = (\lambda_1, \ldots, \lambda_n) \in R^n. \tag{1.3}$$

Let S^n be the set of all $n \times n$ real symmetric matrix. For $S \in S^n$, let $\lambda(S) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the eigenvalues of S. We use the same notion σ_k to define the $k-hessian$ operator as

$$\sigma_k(S) = \sigma_k(\lambda(S)). \tag{1.4}$$

We denote $\Gamma_k = \{\lambda \in R^n | \sigma_i(\lambda) > 0, i=1,\ldots,k\}$, which is an open convex cone in R^n. We also denote $\Gamma_k = \{S \in S^n | \lambda(S) \in \Gamma_k\} = \{S \in S^n | \sigma_i(S) > 0, i=1,\ldots,k\}$ if there is no confusion. It is well known that the $k-hessian$ operator σ_k is elliptic with respect to S in Γ_k and

$\sigma_k^{\frac{1}{k}}$ is concave in Γ_k. For our use here, we only consider the $k=2$ case and we define the following concept of $2-admissible$ functions similar to that of Caffarelli-Nirenberg-Spruck [7].

Definition 1.1. Let $\Omega \subset R^3$ be a domain, a function $u \in C^2(\Omega)$ is said to be a $2-admissible$ function in Ω if $(W_{ij}(D^2u)(x)) \in \Gamma_2$ for every $x \in \Omega$, where $(W_{ij}(D^2u)(x))$ is defined as in (1.2). We denote the set of all C^2 $2-admissible$ functions in Ω by Θ, i.e. $\Theta = \{u \in C^2(\Omega) | (W_{ij}(D^2u)(x)) \in \Gamma_2, x \in \Omega\}$.

Remark 1.1. (1) If u is a $2-admissible$ function, then $F(D^2u) = \sigma_2(W_{ij}(D^2u))$ is elliptic in u. In fact, by a straightforward calculation, we know $F(D^2u) = \sigma_1(D^2u)^2 + \sigma_2(D^2u)$. So, by the chain rule, we have

$$\left(\frac{\partial F}{\partial u_{ij}}\right) = \begin{bmatrix} \frac{\partial \sigma_2}{\partial W_{11}} + \frac{\partial \sigma_2}{\partial W_{22}} & \frac{\partial \sigma_2}{\partial W_{32}} & -\frac{\partial \sigma_2}{\partial W_{31}} \\ \frac{\partial \sigma_2}{\partial W_{23}} & \frac{\partial \sigma_2}{\partial W_{11}} + \frac{\partial \sigma_2}{\partial W_{33}} & \frac{\partial \sigma_2}{\partial W_{21}} \\ -\frac{\partial \sigma_2}{\partial W_{13}} & \frac{\partial \sigma_2}{\partial W_{12}} & \frac{\partial \sigma_2}{\partial W_{22}} + \frac{\partial \sigma_2}{\partial W_{33}} \end{bmatrix}. \tag{1.5}$$

If we denote λ_i' the eigenvalues of $(\frac{\partial \sigma_2}{\partial W_{ij}})$, $(i=1,2,3)$, then by an orthogonal transformation, we know $\lambda_1' + \lambda_2'$, $\lambda_1' + \lambda_3'$, $\lambda_2' + \lambda_3'$ are the eigenvalues of $(\frac{\partial F}{\partial u_{ij}})$. Since u is $2-admissible$, $(W_{ij}(D^2u)) \in \Gamma_2$, we have $(\frac{\partial \sigma_2}{\partial W_{ij}}) > 0$ in Ω, i.e. $\lambda_i' > 0$, so are $\lambda_1' + \lambda_2'$, $\lambda_1' + \lambda_3'$, $\lambda_2' + \lambda_3'$, that is $(\frac{\partial F}{\partial u_{ij}}) > 0$ in Ω.

(2) If u is a $2-admissible$ function, then $G(D^2u) = F^{\frac{1}{2}}(D^2u)$ is concave with respect to D^2u. In fact, we know $\sigma_2^{\frac{1}{2}}(W_{ij})$ is concave with respect to (W_{ij}) because $(W_{ij}(D^2u)) \in \Gamma_2$. Since $W_{ij}(D^2u)$ is linear in u_{ij}, we know $G(D^2u)$ is concave with respect to u_{ij}.

In the following, we will call a $2-admissible$ function an *admissible* one for short. Moreover, a solution to (1.1) is called an *admissible* solution if it is *admissible* in Ω.

Now, we can state our results. Our main purpose here is to derive some convexity for the solution of (1.1). But before doing that, we need first get the existence and uniqueness result for (1.1).

Theorem 1.1. *Let $\Omega \subset R^3$ be a bounded domain in R^3 with smooth boundary. Then, there exists a positive constant λ_1, which depends only on Ω, so that*

(1) *(1.1) possesses a negative admissible solution $\psi \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ for $\lambda = \lambda_1$,*

(2) *if $(\lambda^\star, \psi^\star) \in [0, +\infty) \times (C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}))$ is another solution of (1.1), then $\lambda^\star = \lambda_1, \psi^\star = \alpha\psi_1$ for some positive constant α,*

(3) *if $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.*

Then, we get the convexity result for (1.1).

Theorem 1.2. *Let $\Omega \subset R^3$ be a strictly convex bounded domain in R^3 with smooth boundary. Let $u \in C^\infty(\Omega) \bigcap C^{1,1}(\overline{\Omega})$ be an admissible solution of (1.1), then $-\log(-u)$ is strictly convex in Ω.*

The plan of the paper is as follows. In Section 2, we derive a prior estimate for a general class of equations which will be used in the proof of Theorem 1.1. We finish the proof of Theorem 1.1 in Section 3. Then, we proved the constant rank theorem in Section 4, which is essential to derive the convexity results. In Section 5, we use the deformation technique together with the constant rank theorem to get the proof of Theorem 1.2.

2 C^1 and C^2 estimates

Firstly, we derive the existence and uniqueness result for (1.1). Our method is similar to that of Wang [33]. To do so, we need first get a prior estimate for the following Dirichlet problem.
Consider the Dirichlet problem

$$\begin{cases} F(D^2 u) = f(x,u) & in \ \Omega, \\ u = \varphi & on \ \partial\Omega. \end{cases} \tag{2.1}$$

where $F(D^2 u) = \sigma_2(W_{ij}(D^2 u)) = \sigma_1^2(D^2 u) + \sigma_2(D^2 u)$, $(W_{ij}(D^2 u))$ is defined as in (1.2). Suppose that $f(x,u) \in C^{1,1}(\overline{\Omega} \times R), f \geq 0$, and $\varphi \in C^{3,1}(\partial\Omega)$, we first derive the following a prior C^1 estimate for the solution of (2.1).

Theorem 2.1. *Let Ω be a bounded domain in R^3 with smooth boundary. If $u \in C^3(\Omega) \bigcap C^1(\overline{\Omega})$ is an admissible solution of (2.1), then*

$$\sup_\Omega |Du| \leq M_1, \tag{2.2}$$

where M_1 depends only on $\Omega, \|\varphi\|_{C^3(\partial\Omega)}, |f|_{C^0}, |D_x f|_{C^0}, |D_u f|_{C^0}$ and $M_0 = \sup_{\overline{\Omega}} |u|$, where the C^0 norm of $f, D_x f, D_u f$ are taken over $\overline{\Omega} \times [-M_0, M_0]$.

Proof. We first extend φ harmonically into Ω, which is denoted by $\widetilde{\varphi}$. Since u is *admissible*, $(W_{ij})(x) \in \Gamma_2, x \in \Omega$, we have $\triangle u = \frac{1}{2}\sigma_1(W_{ij}(D^2 u)) > 0$ in Ω, i.e. u is subharmonic in Ω. So, from the comparison principle, we know

$$u \leq \widetilde{\varphi} \quad in \ \Omega. \tag{2.3}$$

Then, for any point $x_0 \in \partial\Omega$, we may assume that x_0 is the origin and the positive x_3 axis is the interior normal there and x_i $(1 \leq i \leq 2)$ are the principal directions of $\partial\Omega$ at x_0. So, $\partial\Omega$ can locally be represented as

$$x_3 = \rho(x') = \frac{1}{2} \sum_{1 \leq i \leq 2} \kappa_i x_i^2 + O(|x'|^3), \tag{2.4}$$

where $\kappa_i(1 \leq i \leq 2)$ are the principal curvatures of $\partial\Omega$ at x_0, $x' = (x_1, x_2)$. Near $\partial\Omega$, let $d(x)$ denote the distance function to $\partial\Omega$. At the origin we have $(d_{ij}) = diag(-\kappa_1, -\kappa_2, 0)$. For large t, we consider $v = \frac{1}{t}(e^{-td} - 1)$, then at the origin

$$(v_{ij}) = (-d_{ij} + td_id_j) = diag(\kappa_1, \kappa_2, t). \tag{2.5}$$

From the definition of $(W_{ij}(D^2v))$, we know

$$(W_{ij}(D^2v)) = diag(t + \kappa_1, t + \kappa_2, \kappa_1 + \kappa_2). \tag{2.6}$$

So $(W_{ij}(D^2v)) \in \Gamma_2$ for t large enough. Following the arguments in Caffarelli-Nirenberg-Spruck [7], we can get a sub-solution $\underline{u}$, such that $\underline{u}$ is *admissible* in $\overline{\Omega}$ and

$$\begin{cases} F(D^2\underline{u}) \geq \sup\limits_{(x,u) \in \overline{\Omega} \times [-M_0, M_0]} f(x, u) & in \ \Omega, \\ \underline{u} = \varphi & on \ \partial\Omega. \end{cases} \tag{2.7}$$

Since both $\underline{u}$ and u are *admissible* in Ω, F is elliptic. So, we can use the comparison principle to deduce that

$$\underline{u} \leq u \qquad in \ \Omega. \tag{2.8}$$

Combining (2.3) and (2.8) and the fact that $\underline{u} = u = \widetilde{\varphi} = \varphi$ on $\partial\Omega$, we get that

$$|Du| \leq C \qquad on \ \partial\Omega. \tag{2.9}$$

Next, we deduce the global gradient estimate.

Let $G(x, \xi) = q(u)u_\xi$, where $q(u) = (2M_0 + 1 - u)^{-\frac{1}{2}}$, u_ξ denotes differentiation in the direction ξ. Suppose $G(x, \xi)$ attains its maximum in $\overline{\Omega}$ at x_1 with $\xi = e_1$. If $x_1 \in \partial\Omega$, then by (2.9), we have done. If $x_1 \in \Omega$, we have $u_1(x_1) = |\nabla u|(x_1)$ and $u_i(x_1) = 0$ for $2 \leq i \leq 3$. We may assume $u_1(x_1) > 0$, or there is nothing to prove. All the following calculations are carried out at x_1. So, we have

$$0 = (\log G)_i = \frac{q'}{q}u_i + \frac{u_{1i}}{u_1}, \quad i = 1, 2, 3, \tag{2.10}$$

$$(\log G)_{ij} = \frac{q''}{q}u_iu_j + \frac{q'}{q}u_{ij} - \frac{(q')^2}{q^2}u_iu_j + \frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2}, \quad i, j = 1, 2, 3. \tag{2.11}$$

From (2.10), we know

$$u_{11} = -\frac{q'}{q}u_1^2 \text{ and } u_{1i} = 0, \quad i = 2, 3. \tag{2.12}$$

Rotate the x_2, x_3 axis such that $\{u_{ij}\}_{2 \leq i, j \leq 3}$ is diagonal at x_1. Then we know $\{u_{ij}\}_{1 \leq i, j \leq 3}$ and $(\frac{\partial F}{\partial u_{ij}})$ are both diagonal at x_1. From Remark 1.1, we know $(F^{ij}) := (\frac{\partial F}{\partial u_{ij}})$ is positive

definite in Ω. Thus

$$
\begin{aligned}
0 &\geq F^{ii}(\log G)ii \\
&= [\frac{q''}{q} - (\frac{q'}{q})^2]F^{ii}u_i^2 + \frac{q'}{q}F^{ii}u_{ii} + \frac{F^{ii}u_{ii1}}{u_1} - \frac{F^{ii}u_{1i}^2}{u_1^2} \\
&= [\frac{q''}{q} - (\frac{q'}{q})^2]F^{ii}u_i^2 + \frac{q'}{q}2F + \frac{f(x,u)_1}{u_1} - F^{11}(\frac{q'}{q})^2\frac{u_1^4}{u_1^2} \\
&\geq [\frac{q''}{q} - 2(\frac{q'}{q})^2]F^{11}u_1^2 + \frac{q'}{q}2f + \frac{f_1}{u_1} + f_u,
\end{aligned}
\tag{2.13}
$$

where we have used the fact that $F(D^2u)$ is homogeneous of degree 2 (so we have $F^{ij}u_{ij} = 2F$). Since

$$
\left[\frac{q''}{q} - 2(\frac{q'}{q})^2\right] \geq \frac{1}{36(M_0+1)^2},
\tag{2.14}
$$

we only need to estimate F^{11} to get an upper bound for $u_1(x_1)$. In fact, since

$$
F^{11} = \frac{\partial\sigma_2}{\partial u_{11}} + 2\sigma_1 = u_{22} + u_{33} + 2\sigma_1,
\tag{2.15}
$$

we have

$$
\begin{aligned}
0 &\leq f(x,u) = F(D^2u) = \sigma_2(D^2u) + \sigma_1^2(D^2u) \\
&= F^{11}u_{11} + \sigma_2(D^2u|1) + \sigma_1^2(D^2u) - 2\sigma_1 u_{11} \\
&= F^{11}u_{11} + u_{22}u_{33} + \sigma_1(u_{22} + u_{33} - u_{11}) \\
&\leq F^{11}u_{11} + \frac{(u_{22}+u_{33})^2}{4} + 2\sigma_1(u_{22} + u_{33}) \\
&\leq F^{11}u_{11} + \frac{1}{2}(F^{11})^2 = -F^{11}\frac{q'}{q}u_1^2 + \frac{1}{2}(F^{11})^2,
\end{aligned}
\tag{2.16}
$$

where we use the cauchy inequality and the fact that $\triangle u > 0$, i.e. $-u_{11} < u_{22} + u_{33}$ for the second "$\leq$" and we use (2.15) for the last one. It then follows from (2.16) that (notice that $F^{11} > 0$ since $(F^{ij}) > 0$)

$$
F^{11} \geq 2\frac{q'}{q}u_1^2 \geq \frac{1}{6(M_0+1)}u_1^2.
\tag{2.17}
$$

Inserting (2.14) and (2.17) into (2.13), we get an upper bound for $u_1(x_1) = |\nabla u|(x_1)$, which also implies a global upper bound for $|\nabla u|$. $\qquad\square$

Now, we derive the a prior C^2 estimate of (2.1) when $\varphi \equiv 0$ on $\partial\Omega$.

Theorem 2.2. *Let Ω be a bounded domain in R^3 with smooth boundary. Suppose $\varphi = 0$ on $\partial\Omega$ and*

$$D_x(f^{\frac{1}{2}}), D_u(f^{\frac{1}{2}}) \in C^1(\overline{\Omega} \times R), \tag{2.18a}$$

$$|D_x(f^{\frac{1}{2}})|_{C^0}, |D_u(f^{\frac{1}{2}})|_{C^0} \leq C_1 < +\infty, \tag{2.18b}$$

$$|D^2_{xx}(f^{\frac{1}{2}})|_{C^0}, |D^2_{xu}(f^{\frac{1}{2}})|_{C^0}, |D^2_{uu}(f^{\frac{1}{2}})|_{C^0} \leq C_2 < +\infty. \tag{2.18c}$$

Then if $u \in C^{3,1}(\Omega) \cap C^3(\overline{\Omega})$ is an admissible solution of (2.1), we have

$$\sup_{\Omega} |D^2 u| \leq M_2, \tag{2.19}$$

where M_2 depends only on $\Omega, \|f\|_{C^0}, C_1, C_2, \|\nabla u\|_{C^0(\overline{\Omega})}$, where the C^0 norms related to f are taken over $\overline{\Omega} \times [-M_0, M_0]$.

Proof. The outline of our proof is similar to that of Theorem 3.2 in [33]. Rewrite the equation in (2.1) as

$$G(D^2 u) = \sigma_2^{\frac{1}{2}}(W_{ij}(D^2 u)) = g(x, u) \qquad in \ \Omega, \tag{2.20}$$

where $g = f^{\frac{1}{2}}$, and denote

$$G_{ij}(r) = \frac{\partial G}{\partial r_{ij}}(r), \quad G_{ij,st}(r) = \frac{\partial^2 G}{\partial r_{st} \partial r_{ij}}(r), \tag{2.21}$$

then we have

$$G_{ij} D_{ij} u_k = D_k g, \tag{2.22a}$$

$$G_{ij} D_{ij} u_{kl} = -G_{ij,st} u_{ijk} u_{stl} + D_{kl} g. \tag{2.22b}$$

Since u is *admissible*, $G(D^2 u)$ is concave with respect to u_{ij} by Remark 1.1 (2). So from (2.22) and (2.18), we know

$$G_{ij} D_{ij}(\triangle u) \geq \triangle g \geq g_u \triangle u - C. \tag{2.23}$$

Observing that

$$G_{ij}(D^2 u) = \frac{1}{2} \sigma_2^{-\frac{1}{2}}(W_{ij}) \frac{\partial \sigma_2(W_{ij})}{\partial u_{ij}}.$$

From Remark 1.1 (1), we know that if we denote λ'_i the eigenvalues of $\left(\frac{\partial \sigma_2(W_{ij})}{\partial W_{ij}} \right)$ $(i=1,2,3)$, then $\lambda'_1 + \lambda'_2$, $\lambda'_1 + \lambda'_3$, $\lambda'_2 + \lambda'_3$ are the eigenvalues of $\left(\frac{\partial \sigma_2(W_{ij})}{\partial u_{ij}} \right)$ and all positive. So, by the

Newton−MacLaurin inequality, we have

$$det\left(\frac{\partial\sigma_2(W_{ij})}{\partial u_{ij}}\right) = (\lambda_1'+\lambda_2')(\lambda_1'+\lambda_3')(\lambda_2'+\lambda_3') = \sigma_1(\lambda')\sigma_2(\lambda') - \sigma_3(\lambda')$$

$$\geq 8\sigma_3(\lambda') = 8det\left(\frac{\partial\sigma_2(W_{ij})}{\partial W_{ij}}\right) \geq 8C[\sigma_2(W_{ij})]^{\frac{3}{2}},$$

where $C=(\frac{2}{3})^3 3^{\frac{3}{2}}$ (see Prop 2.2 in [33]). So we have $det(G_{ij})\geq C$. Applying the Alexandrov maximum principle to (2.23), we obtain (since $\triangle u\geq 0$)

$$\sup_\Omega \triangle u \leq \sup_{\partial\Omega}\triangle u + C\|1+\triangle u\|_{L^3(\Omega)}$$

$$\leq \sup_{\partial\Omega}\triangle u + C\sup_\Omega(1+\triangle u)^{\frac{2}{3}}\left(\int_\Omega(1+\triangle u)dx\right)^{\frac{1}{3}} \tag{2.24}$$

$$\leq \sup_{\partial\Omega}\triangle u + C\sup_\Omega(1+\triangle u)^{\frac{2}{3}},$$

where we use the divergence theorem and the C^1 estimate in (2.2) for the last "$\leq$". So, we have

$$\sup_\Omega \triangle u \leq C(1+\sup_{\partial\Omega}\triangle u). \tag{2.25}$$

For any fixed $x\in\Omega$, by an orthogonal transformation of coordinates we may suppose $(u_{ij})(x)$ is diagonal, then $(W_{ij}(D^2u)(x))$ is also diagonal. Since (W_{ij}) is in Γ_2, we have $W_{ii}+W_{jj}>0$, $i,j=1,2,3$, $i\neq j$, i.e.

$$u_{ii} > -\triangle u, \quad i=1,2,3. \tag{2.26}$$

On the other hand, we have

$$W_{ii}+W_{jj} \leq (W_{11}+W_{22})+(W_{11}+W_{33})+(W_{22}+W_{33}) = 4\triangle u, \quad i,j=1,2,3, i\neq j,$$

i.e. $\triangle u+u_{ii}\leq 4\triangle u$, $i=1,2,3$ or

$$u_{ii} \leq 3\triangle u, \quad i=1,2,3. \tag{2.27}$$

In other words, we have $|u_{ii}| \leq 3\triangle u, i=1,2,3$. Since i is arbitrary and $\triangle u$ is invariant under rotation, we can rotate the coordinates to get

$$|D^2u| \leq C|\triangle u| = C\triangle u \qquad in\ \Omega. \tag{2.28}$$

Combining (2.28) and (2.25), we get

$$\sup_\Omega|D^2u| \leq C(1+\sup_{\partial\Omega}|D^2u|). \tag{2.29}$$

It remains to estimate $|D^2 u|$ on the boundary. For any given $x \in \partial\Omega$, we may suppose it is the origin and $\partial\Omega$ can locally be represented as in (2.4) near the origin. Since $u = \varphi = 0$ on $\partial\Omega$, we have

$$u_{\alpha\beta}(0) = -u_3 \kappa_\alpha \delta_{\alpha\beta}, \qquad 1 \leq \alpha, \beta \leq 2. \tag{2.30}$$

Let $\widetilde{w} = \rho(x') - x_3 - \delta |x'|^2 + K x_3^2$, then it is easy to check that $\widetilde{w}$ is admissible in a small neighborhood of x in Ω since $(W_{ij}(D^2 \widetilde{w}))$ has eigenvalues $\kappa_1 + \kappa_2 - 4\delta$, $2K + \kappa_1 - 2\delta$, $2K + \kappa_2 - 2\delta$ at x. So $G(D^2 \widetilde{w})$ is concave with respect to $D^2 \widetilde{w}$, we can use the similar arguments as in Caffarelli-Nirenberg-Spruck [7] or Wang [34] to show that

$$|u_{\alpha 3}(0)| \leq C, \qquad 1 \leq \alpha \leq 2. \tag{2.31}$$

To estimate $|u_{33}(0)|$, we first note that u is subharmonic in Ω (and in $\overline{\Omega}$ by continuity). This together with (2.30) implies a lower bound for $u_{33}(0)$. On the other hand, by (2.30), we have

$$(W_{ij}(D^2 u)(0)) = \begin{bmatrix} -u_3(\kappa_1 + \kappa_2) & u_{32} & -u_{31} \\ u_{23} & -u_3 \kappa_1 + u_{33} & 0 \\ -u_{13} & 0 & -u_3 \kappa_2 + u_{33} \end{bmatrix}. \tag{2.32}$$

So we have

$$\begin{aligned} C \geq f(x,0) &= \sigma_2(W_{ij}(D^2 u)(0)) \\ &= u_{33}^2 - 3u_{33}u_3(\kappa_1 + \kappa_2) + u_3^2(\kappa_1^2 + \kappa_2^2 + 3\kappa_1\kappa_2) - u_{13}^2 - u_{23}^2. \end{aligned} \tag{2.33}$$

Since u_3, u_{13}, u_{23} are all bounded, this implies that $u_{33}(0) \leq C$.

$\square$

Remark 2.1. (1) If $f \in C^{1,1}(\overline{\Omega} \times R)$ is strictly positive in $\overline{\Omega}$, then by Theorem 2.1, 2.2 and the standard elliptic regularity theory we have

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C. \tag{2.34}$$

(2) If f satisfies (2.18) and is strictly positive in Ω, then by the interior Hölder estimates for second order derivatives, we have $\|u\|_{C^{2,\alpha}(\overline{\Omega_1})} \leq C$ for any $\Omega_1 \subset\subset \Omega$. Hence $u \in C^{2,\alpha}(\Omega) \cap C^{1,1}(\overline{\Omega})$.

(3) Thanks to the special structure of $(W_{ij}(D^2 u))$ with respect to $D^2 u$ in the case $k = 2$, $n = 3$ (i.e. the eigenvalues of $(W_{ij}(D^2 u))$ are the "choosing sum" of that of $D^2 u$), we did not make any assumptions on the convexity of $\partial\Omega$ to get the boundary C^1 and C^2 estimates here. However, for general k and n, we may need $\partial\Omega$ to satisfy some convexity condition which was studied in Harvey-Lawson [18–20] to get these estimates.

From Theorem 2.1 and 2.2 and Remark 2.1.(2), we therefore obtain the existence result for the homogenous Dirichlet problem by the method of super and sub solution.

Theorem 2.3. *Suppose $f \in C^{1,1}(\overline{\Omega} \times R)$ satisfies (2.18) and*

$$f(x,u) > 0 \qquad for \ u < 0. \tag{2.35}$$

If there are a sub-solution w and a super-solution v to

$$\begin{cases} F(D^2 u) = f(x,u) & in \ \Omega, \\ u = 0 & on \ \partial\Omega. \end{cases} \tag{2.36}$$

then (2.36) possesses a solution $u \in C^{2,\alpha}(\Omega) \cap C^{1,1}(\overline{\Omega})$ with $v \geq u \geq w$.

A function is said to be a sub-solution (resp. super-solution) of (2.36), if

$$\begin{cases} F(D^2 u) \geq (resp. \leq) f(x,u) & in \ \Omega, \\ u = 0 & on \ \partial\Omega. \end{cases} \tag{2.37}$$

Proof. This can be derived from the uniform $C^{2,\alpha}$ estimate in Remark 2.1 and the standard super-sub solution method (see Wang [32] or Aubin [2]). □

3   Proof of Theorem 1.1

In this section, we will use the results in Section 2 to prove Theorem 1.1. First, let's state a lemma which will be used in the proof. This lemma was got by Geng-Yu-Qu [12].

Lemma 3.1. *Suppose $u \in C^2(\overline{\Omega})$ and $\triangle u > 0$ in Ω. If there exists $x_0 \in \Omega$, positive constants R and δ such that*

$$\triangle u \geq \delta, \qquad x \in B_R(x_0) \subset \Omega, \tag{3.1}$$

then

$$\frac{\partial u}{\partial \nu}(x) \geq \alpha, \qquad x \in \partial\Omega, \tag{3.2}$$

where ν is the unit outer normal of $\partial\Omega$, $\alpha > 0$ depends only on Ω, R, δ and x_0's position in Ω.

We now prove Theorem 1.1. The proof is similar to that in [12, 25, 33].

Proof. (1) Let's first introduce a set Λ and a nonnegative constant λ_1 by

$$\Lambda = \left\{ \lambda \geq 0 \mid there \ is \ a \ solution \ u_\lambda \in C^2(\overline{\Omega}) \ of \ (3.4) \right\}$$

$$\lambda_1 = \sup \Lambda, \tag{3.3}$$

where (3.4) is defined as

$$\begin{cases} F(D^2 u) = (1 - \lambda u)^2 & in \ \Omega, \\ u = 0 & on \ \partial\Omega. \end{cases} \tag{3.4}$$

and $F(D^2u) = \sigma_2(W_{ij}(D^2u))$ is the same as in Section 2. We will show that λ_1 is positive and finite, and that as $\lambda \to \lambda_1$, $\frac{u_\lambda}{\|u_\lambda\|_{C^0}}$ converges to a solution of (1.1) with $\lambda = \lambda_1$ there.

We know from Remark 2.1 (3) and Caffarelli-Nirenberg-Spruck [7] that

$$\begin{cases} F(D^2u) = 1 & in \ \Omega, \\ u = 0 & on \ \partial\Omega. \end{cases} \tag{3.5}$$

admits an *admissible* solution η. Then,

$$F(D^2(2\eta)) = 2^2 \geq (1 - 2\lambda\eta)^2 \ \ for \ \lambda \in \left(0, \frac{1}{2\sup_\Omega |\eta|}\right).$$

From Theorem 2.3 and Remark 2.1, we obtain a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (3.4) for $\lambda \in (0, \frac{1}{2\sup_\Omega |\eta|})$. Hence $\lambda_1 > 0$.

To see that λ_1 is finite, just observe that if (λ, u) is a solution of (3.4), then we have $(W_{ij}(D^2u)) \in \overline{\Gamma}_2$. So u is subharmonic in Ω and $u \leq 0$ in Ω. So, by (3.4), we have $\sigma_2(W_{ij}(D^2u)) > 0$ in Ω and by the *Newton−Macularin* inequality, we have $\triangle u > 0$ in Ω, which implies that $u < 0$ in Ω. By *Newton−Macularin* inequality, we have

$$\triangle u = \frac{1}{2}\sigma_1(W_{ij}(D^2u)) \geq C\sigma_2^{\frac{1}{2}}(W_{ij}(D^2u)) = C(1 - \lambda u). \tag{3.6}$$

Let $\overline{\lambda}$ be the first eigenvalue of $-\frac{\triangle}{C}$ with homogenous Dirichlet boundary value and ψ be a positive eigenfunction corresponding to $\overline{\lambda}$. Multiplying (3.6) by ψ and integrating by parts, we have

$$\int_\Omega (\lambda u - 1)\psi dx \geq \int_\Omega -\frac{\triangle u}{C}\psi dx = \int_\Omega -\frac{\triangle \psi}{C}u dx = \overline{\lambda}\int_\Omega \psi u dx, \tag{3.7}$$

i.e.

$$(\overline{\lambda} - \lambda)\int_\Omega \psi u dx \leq -\int_\Omega \psi dx < 0. \tag{3.8}$$

Since $\psi u < 0$ in Ω, this implies that $\lambda_1 \leq \overline{\lambda} < +\infty$.

From the definition of λ_1, we see that for any $\lambda_0 \in [0, \lambda_1)$, there exists λ_2, $\lambda_0 \leq \lambda_2 < \lambda_1$ s.t. $\lambda_2 \in \Lambda$. So u_{λ_2} is a sub-solution of (3.4) for $\lambda = \lambda_0$. By Theorem 2.3, we know $\lambda_0 \in \Lambda$, i.e. $[0, \lambda_1) \subset \Lambda$. Next, we claim $\|u_\lambda\|_{C^0}$ tends to $+\infty$ as $\lambda \to \lambda_1$. In deed, if it is not the case, then there exists a sequence $\{\gamma_i\}_{i=1}^{+\infty} \subset \Lambda$, s.t $\gamma_i \to \lambda_1$ with $\|u_{\gamma_i}\|_{C^0} \leq C$ where C is independent of i. By Theorem 2.1 and Theorem 2.2, we know $\|u_{\gamma_i}\|_{C^2(\overline{\Omega})} \leq C$ with C independent of i. Since $f(x, u_{\gamma_i}) \geq 1$ for $x \in \overline{\Omega}$, by Remark 2.1, we know $\|u_{\gamma_i}\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ with C independent of i. So there is a subsequence of $\{u_{\gamma_i}\}$ that converges to some function $u^\star$ in $C^2(\overline{\Omega})$. Clearly, $(\lambda_1, u^\star)$ is a solution of (3.4) and $u^\star \not\equiv 0$. Take $\delta > 0$, such that $\delta < 1/\|u^\star\|_{C^0}$ and set $\underline{u} = (1 - \delta\|u^\star\|_{C^0})^{-1}u^\star$, then $\underline{u} = 0$ on $\partial\Omega$ and since

$$\frac{1 - \lambda_1 u^\star}{1 - \delta\|u^\star\|_{C^0}} \geq 1 - (\lambda_1 + \delta)\underline{u},$$

we have

$$F(D^2\underline{u}) = (1-\delta\|u^\star\|_{C^0})^{-2}(1-\lambda_1 u^\star)^2 \geq (1-(\lambda_1+\delta)\underline{u})^2. \tag{3.9}$$

Hence by Theorem 2.3, we have $\lambda_1+\delta\in\Lambda$. This contradicts to the definition of λ_1. So the claim is true.

Denote $v_\lambda = u_\lambda/\|u_\lambda\|_{C^0}$, then $v_\lambda\in[-1,0]$ and satisfies

$$\begin{cases} F(D^2 v) = (\|u_\lambda\|_{C^0}^{-1} - \lambda v)^2 & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{3.10}$$

Since $\|v_\lambda\|_{C^0}$ is uniformly bounded, by Theorem 2.1 and Theorem 2.2 we have

$$\|v_\lambda\|_{C^2(\overline{\Omega})} \leq C, \tag{3.11}$$

where C is independent of λ. Hence there is a subsequence of $\{v_\lambda\}$, we denote it by $v_{\mu_j} = v_j$, such that ($\mu_j > \frac{\lambda_1}{2} > 0$ and) v_j tends to some function $v_0\in C^0(\overline{\Omega})$ uniformly in $\overline{\Omega}$. Since $\|v_j\|_{C^0(\overline{\Omega})}\equiv 1$, we have $\|v_0\|_{C^0(\overline{\Omega})} = 1$. Since $v_0 = 0$ on $\partial\Omega$, there exists $x_0\in\Omega$, $R>0$ and $\delta > 0$ such that $v_0(x_0) = -1, B_{2R}(x_0)\subset\Omega$ and

$$v_0(x) \leq -2\delta, \quad x\in B_{2R}(x_0). \tag{3.12}$$

Thus there is N, such that

$$v_j(x) \leq -\delta, \quad j\geq N, x\in B_R(x_0). \tag{3.13}$$

On the other hand, since $v_j < 0$ in Ω and $(W_{ij}(D^2 v_j))\in\Gamma_2$, we have

$$\triangle v_j \geq C\sigma_2^{\frac{1}{2}}(W_{ij}(D^2 v_j)) \geq -\mu_j v_j > 0 \ \ in\ \Omega$$

and $\triangle v_j \geq \frac{\lambda_1}{2}\delta$ in $B_R(x_0), j\geq N$. Set $B_R = B_R(x_0), r = |x-x_0|(x\in R^3)$ and let

$$w_j(x) = v_j(x) - \varepsilon\varphi(r)v(x), \quad j\geq N, x\in\overline{\Omega}, \tag{3.14}$$

where $\varphi\in C^\infty[0,+\infty)$ satisfying $0\leq\varphi(r)\leq 1; |\varphi'|,|\varphi''|\leq M$ and

$$\varphi(r) = \begin{cases} 0 & r\leq\dfrac{R}{2}, \\ 1 & r\geq R, \end{cases} \tag{3.15}$$

v is the unique solution of

$$\begin{cases} \triangle u = 0 & in\ \Omega - \overline{B}_{\frac{R}{2}}, \\ u = -1 & on\ \partial B_{\frac{R}{2}}, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{3.16}$$

Then, similar to the proof of Lemma 3.1 in [12], we have $\triangle w_j > 0$ in Ω for $\varepsilon > 0$ small enough. So by the maximum principle, $w_j < 0$ in Ω, hence together with (3.13), we have

$$v_j(x) \leq \begin{cases} \varepsilon v(x), & r = |x - x_0| > R, \\ -\delta, & r = |x - x_0| \leq R. \end{cases} \tag{3.17}$$

Therefore, for any open set $\Omega' \subset\subset \Omega$, there is $\sigma = \sigma(\Omega') > 0$ such that

$$v_j(x) \leq -\sigma, \quad j \geq N, \ x \in \Omega'. \tag{3.18}$$

So, from Remark 2.1 (2), there is $C = C(\sigma, \Omega') > 0$ such that

$$\|v_j\|_{C^{2+\alpha}(\Omega')} \leq C, \tag{3.19}$$

where $0 < \alpha < 1$, C is independent of j. Hence there is a subsequence of $\{v_j\}$ converges to some function $\psi_1 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$, which is a solution of (1.1) with $\lambda = \lambda_1$, $(W_{ij}(D^2\psi_1)) \in \overline{\Gamma}_2$ in Ω. On the other hand, we have $\psi_1 < 0$ in Ω by (3.18). So by (1.1) we have $\sigma_2(W_{ij}(D^2\psi_1)) > 0$ in Ω and together with the *Newton $-$ Macularin* inequality, we have $(W_{ij}(D^2\psi_1)) \in \Gamma_2$ in Ω, i.e. ψ_1 is *admissible*.

(2) If $(\lambda_1, u_1), (\lambda_2, u_2) \in [0, +\infty) \times (C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}))$ are two pairs solutions of (1.1) with u_i admissible in Ω, $i = 1, 2$. Let $L_i = G^{ij}(D^2 u_i)\partial_{ij}$, then

$$L_i u_i = G(D^2 u_i) = -\lambda_i u_i \quad in \ \Omega, \ i = 1, 2. \tag{3.20}$$

By the properties of the first eigenvalue of non-divergence form elliptic operators (see Evans [9] chapter 6.5 Theorem 3), we have

$$\lambda_i = Re(\lambda_i) \geq \lambda_1(-L_i) = -\lambda_1(L_i), \quad i = 1, 2. \tag{3.21}$$

where $\lambda_1(L_i)$ is the first eigenvalue of L_i.

On the other hand, since $(W_{ij}(D^2 u_i)) \in \Gamma_2$ in Ω. By the concavity and homogeneity of G, we have

$$L_1 u_2 \geq G(u_1 + u_2) - G(u_1) \geq G(u_1) + G(u_2) - G(u_1) = G(u_2) = -\lambda_2 u_2 \quad in \ \Omega. \tag{3.22}$$

By the characterization of the first eigenvalue in Nirenberg [3], we know

$$-\lambda_1(L_1) = sup\{\lambda \in R | \exists \phi > 0 \ in \ \Omega, s.t. \ L_1\phi + \lambda\phi \leq 0 \ in \ \Omega\}, \tag{3.23}$$

where ϕ is in $W_{loc}^{2,p}(\Omega)$. Since $-u_2 > 0$ in Ω and $-u_2 \in C^\infty(\Omega) \subset W_{loc}^{2,p}(\Omega)$, we have by (3.22) and (3.23) that

$$\lambda_2 \leq -\lambda_1(L_1), \tag{3.24}$$

which together with (3.21) imply that $\lambda_2 \leq \lambda_1$. Similarly, $\lambda_1 \leq \lambda_2$. So $\lambda_1 = \lambda_2$ and all the inequality in (3.21), (3.24) should be ″=″ and $\lambda_1 = \lambda_2 = -\lambda_1(L_1) = -\lambda_1(L_2)$.

Since $(W_{ij}(D^2u_2))\in\Gamma_2$ in Ω, we have $\triangle u_2=\frac{1}{2}\sigma_1(W_{ij}(D^2u_2))>0$ in Ω. By Hopf lemma, we have

$$\frac{\partial u_2}{\partial \nu}(x)>0,\quad x\in\partial\Omega, \tag{3.25}$$

where ν is the unit outer normal of $\partial\Omega$. Hence, there exists $t>0$ small such that $tu_1-u_2\geq 0$ in $\overline{\Omega}$. Let $t_0=sup\{t>0|tu_1-u_2\geq 0\ in\ \overline{\Omega}\}$, then we have $0<t_0<+\infty$ ($t_0<+\infty$ because $u_1<0$ in Ω).

By the definition of t_0, we know $t_0u_1-u_2\geq 0$ in $\overline{\Omega}$. On the other hand, by (3.20) and (3.22), we have

$$-L_1(t_0u_1-u_2)\geq -t_0L_1u_1-\lambda_2u_2=\lambda_1(t_0u_1-u_2)\quad in\ \Omega. \tag{3.26}$$

If $\exists x_0\in\Omega$ such that $(t_0u_1-u_2)(x_0)=0$, then $t_0u_1-u_2$ attains its minimum at $x_0\in\Omega$. Since we have $-L_1(t_0u_1-u_2)\geq 0$ in Ω by (3.26), So $t_0u_1-u_2\equiv 0$ in $\overline{\Omega}$ by the strong maximum principle and we have done in this case.

If $(t_0u_1-u_2)(x)>0,\forall x\in\Omega$, then we have $-L_1(t_0u_1-u_2)>0,\forall x\in\Omega$ by (3.26). By Hopf lemma, we have

$$\frac{\partial(t_0u_1-u_2)}{\partial \nu}(x)<0,\ x\in\partial\Omega. \tag{3.27}$$

So there exists $\varepsilon>0$ such that $t_0u_1-u_2\geq\varepsilon(-u_1)$ in Ω (hence in $\overline{\Omega}$), i.e. $(t_0+\varepsilon)u_1-u_2\geq 0$ in $\overline{\Omega}$. This contradicts to the definition of t_0. So we have $u_2\equiv t_0u_1$ in $\overline{\Omega}$.

(3) Let $\lambda_i=\lambda_1(\Omega_i)$ with the corresponding admissible eigenfunction u_i, $i=1,2$. Since u_2 is *admissible* in Ω_2, it is also *admissible* in Ω_1. So (3.22) holds in Ω_1. By (3.23) and the above argument, we know $\lambda_2\leq -\lambda_1(L_1)=\lambda_1$. $\qquad\square$

4 Constant rank theorem

Let $\Omega\subset R^3$ be a strictly convex bounded domain. We now move to the proof of Theorem 1.2. From Theorem 1.1, we know there exists $(\lambda,u)\in(0,+\infty)\times(C^\infty(\Omega)\cap C^{1,1}(\overline{\Omega}))$ be the unique (up to a positive multiplier) admissible solution of (1.1).

As we have known above, $F(D^2u)=\sigma_2(W_{ij}(D^2u))=\sigma_1^2(D^2u)+\sigma_2(D^2u)$, set $v=-\log(-u)$, then we have

$$u=-e^{-v},\quad u_i=e^{-v}v_i,\quad u_{ij}=e^{-v}(v_{ij}-v_iv_j). \tag{4.1}$$

So, (1.1) is equivalent to

$$\begin{cases} \sigma_2(D^2v)-tr(MD^2v)+(\sigma_1(D^2v)-|Dv|^2)^2=\lambda & in\ \Omega, \\ v\to+\infty & on\ \partial\Omega. \end{cases} \tag{4.2}$$

where $M=M(Dv)=|\nabla v|^2I-DvDv^T$ is a positive semi-definite matrix. To prove the strict convexity of v in Ω, we first derive a constant rank theorem for it.

Lemma 4.1 (Constant Rank Theorem). *Let $\Omega \subset R^3$ be a bounded domain, $u \in C^\infty(\Omega) \bigcap C^{1,1}(\overline{\Omega})$ be an admissible solution of (1.1) (so that $v = -\log(-u)$ satisfies (4.2)) and $D^2 v \geq 0$ in Ω, then $D^2 v$ is of constant rank in Ω.*

Proof. Let r_0 be the smallest rank of $D^2 v$ in Ω. We only need to prove rank$(D^2 v) \equiv r_0$ in Ω. Firstly, as u is *admissible*, $\sigma_1(D^2 u) = \frac{1}{2}\sigma_1(W_{ij}(D^2 u)) > 0$ in Ω. So $\sigma_1(D^2 v) = e^v \sigma_1(D^2 u) + |Dv|^2 > 0$ by (4.1), r_0 can only be 1,2,3. If $r_0 = 3$, then there is nothing to prove. Next, we consider the case when $r_0 = 1, 2$.

Set $P = \sigma_{r_0+1}(D^2 v), \Omega_1 = \{x \in \Omega | P(x) = 0\}$. Since $D^2 v$ is semi-positive in Ω and r_0 is the smallest rank of $D^2 v$ in Ω, we shall prove that

$$\Omega_1 = \Omega. \tag{4.3}$$

So there are exactly r_0 eigenvalues of $(D^2 v)$'s which are positive everywhere in Ω, i.e. rank$(D^2 v) \equiv r_0$ in Ω. First, we know Ω_1 is not empty since $D^2 v$ must attains the smallest rank r_0 at some point in Ω. Note that, Ω_1 is closed in Ω from its definition. Now, let $x_0 \in \Omega_1$, we shall prove that there exists a neighborhood O_1 of x_0 in Ω such that $P \equiv 0$ in O_1, which implies that Ω_1 is open. Since Ω is connected, we must have $\Omega_1 = \Omega$.

Following the notations in [6] and [23], let O_1 be any open set and h, k be functions defined in O_1. Let $y \in O_1$, we say $h(y) \precsim k(y)$ if there exist two positive constants c_1, c_2 such that

$$h(y) \leqslant k(y) + (c_1 P + c_2 |\nabla P|)(y). \tag{4.4}$$

We say $h(y) \sim k(y)$ if both $h(y) \precsim k(y)$ and $k(y) \precsim h(y)$ hold. Next, we say $h \precsim k$ in O_1 if (4.4) holds in O_1 with c_1, c_2 independent of $y \in O_1$. Finally, we say $h \sim k$ in O_1 if we both have $h \precsim k$ and $k \precsim h$ hold in O_1.

Now, let $F = F(Dv, D^2 v) = \sigma_2(D^2 v) - tr(MD^2 v) + (\sigma_1(D^2 v) - |Dv|^2)^2$, and we use the following notations

$$F^{ij} = \frac{\partial F}{\partial v_{ij}}, \quad F^{p_l} = \frac{\partial F}{\partial v_l}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial v_{rs} \partial v_{ij}}, \quad F^{ij,p_l} = \frac{\partial^2 F}{\partial v_l \partial v_{ij}}.$$

Fixing a point $x_0 \in \Omega_1$, we shall prove that

$$F^{ij} P_{ij} \precsim 0 \tag{4.5}$$

in an open neighborhood $O_1 \subset \Omega_1$ of x_0. Then by the strong maximum principle, we know $P \equiv P(x_0) = 0$ in O_1. To prove (4.5), we assume $O_1 \subset\subset \Omega$. We fix a point $x \in O_1$ and rotate the coordinates such that (v_{ij}) is diagonal and $v_{11} \geq v_{22} \geq v_{33}$ at x. We deal with the two cases ($r_0 = 1$ and $r_0 = 2$) separately.

Case 1: $r_0 = 2$. In this case, $P(x) = det(D^2 v(x))$. Since the smallest rank of $D^2 v$ is 2, there exists a positive constant $c_1 > 0$, depending only on $\|v\|_{C^3}$ and dist$(O_1, \partial\Omega)$, such that $v_{11} \geq v_{22} \geq c_1 > 0$ in O_1. In the following, we do the calculations at x with the constant

under control and independent of $x \in O_1$. First, since $D^2 v$ diagonal at x and $v_{11} \geq v_{22} \geq c_1 > 0$ in O_1, we have $P(x) = v_{11} v_{22} v_{33}$ and $0 \leq v_{33} = \frac{P}{v_{11} v_{22}} \leq \frac{P}{c_1^2}$, that is

$$v_{33} \sim P \sim 0. \tag{4.6}$$

Differentiating P with respect to x_i and x_j, we get

$$v_{33i} \sim P_i \sim 0, \tag{4.7}$$
$$P_{ij} \sim v_{11} v_{22} v_{33ij} - 2 v_{22} v_{13i} v_{13j} - 2 v_{11} v_{23i} v_{23j}, \tag{4.8}$$

which implies that

$$F^{ij} P_{ij} \sim v_{11} v_{22} F^{ij} v_{ij33} - 2 v_{22} F^{ij} v_{13i} v_{13j} - 2 v_{11} F^{ij} v_{23i} v_{23j}. \tag{4.9}$$

Now, we compute the partial derivatives of F along v_{ij}, it follows that

$$F^{11} \sim 2 v_{11} + 3 v_{22} - 2 v_1^2 - 3(v_2^2 + v_3^2), \quad F^{22} \sim 2 v_{22} + 3 v_{11} - 2 v_2^2 - 3(v_1^2 + v_3^2), \tag{4.10a}$$
$$F^{12} \sim v_2 v_1 \sim v_1 v_2 \sim F^{21}, \quad F^{11,11} = F^{22,22} = 2, \tag{4.10b}$$
$$F^{11,22} = F^{22,11} = 3, \quad F^{12,21} = F^{21,12} = -1. \tag{4.10c}$$

Next, differentiating (4.2) with respect to x_3, we have

$$F^{p_l} v_{l3} + F^{ij} v_{ij3} = 0. \tag{4.11}$$

Differentiating with respect to x_3 again, we have

$$F^{ij,rs} v_{ij3} v_{rs3} + F^{ij} v_{ij33} \sim 0. \tag{4.12}$$

By (4.12) and (4.10) we know that

$$F^{ij} v_{ij33} \sim -F^{ij,rs} v_{ij3} v_{rs3} \sim -2 v_{113}^2 - 2 v_{223}^2 - 6 v_{113} v_{223} + 2 v_{123}^2. \tag{4.13}$$

On the other hand, by (4.11), we have

$$2 F^{12} v_{123} \sim -F^{11} v_{113} - F^{22} v_{223}, \tag{4.14}$$

which implies that

$$F^{ij} v_{13i} v_{13j} \sim -F^{22} v_{113} v_{223} + F^{22} v_{123}^2, \quad F^{ij} v_{23i} v_{23j} \sim -F^{11} v_{113} v_{223} + F^{11} v_{123}^2. \tag{4.15}$$

Putting (4.13) and (4.15) into (4.9) to substitute corresponding terms, we get

$$F^{ij} P_{ij} \sim 2(v_{11} v_{22} - F^{11} v_{11} - F^{22} v_{22})(v_{123}^2 - v_{113} v_{223}) - 2 v_{11} v_{22}(v_{113} + v_{223})^2. \tag{4.16}$$

On the other hand, since $v_{33} \sim 0$ and $D^2 v$ is diagonal at x, we have

$$\lambda = F(Dv, D^2 v) \tag{4.17}$$
$$\sim v_{11}^2 + v_{22}^2 + 3 v_{11} v_{22} - v_{11}[2 v_1^2 + 3(v_2^2 + v_3^2)] - v_{22}[2 v_2^2 + 3(v_1^2 + v_3^2)] + |Dv|^4.$$

From (4.10) and (4.17), we know

$$v_{11} v_{22} - F^{11} v_{11} - F^{22} v_{22} \sim -\sigma_1^2(D^2 v) - \lambda + |Dv|^4. \tag{4.18}$$

Putting (4.18) into (4.16), we have

$$F^{ij} P_{ij} \sim 2(-\sigma_1^2(D^2 v) - \lambda + |Dv|^4)(v_{123}^2 - v_{113} v_{223}) - 2 v_{11} v_{22}(v_{113} + v_{223})^2. \tag{4.19}$$

Now, we only need to prove $(-\sigma_1^2(D^2 v) - \lambda + |Dv|^4)(v_{123}^2 - v_{113} v_{223}) \lesssim 0 \, in \, O_1$. First, notice that

$$\sigma_1(D^2 v) = e^v \sigma_1(D^2 u) + |Dv|^2 \geq |Dv|^2 \geq 0.$$

So we have

$$-\sigma_1^2(D^2 v) - \lambda + |Dv|^4 \leq -\lambda < 0. \tag{4.20}$$

Next, we know from (4.17) that

$$4 v_1^2 v_2^2 v_{123}^2$$
$$\gtrsim 4[2 v_{11} + 3 v_{22} - 2 v_1^2 - 3(v_2^2 + v_3^2)][2 v_{22} + 3 v_{11} - 2 v_2^2 - 3(v_1^2 + v_3^2)] v_{113} v_{223}$$
$$\sim 4[\frac{13}{3}\lambda + \frac{5}{3}(v_{11} - v_1^2 - \frac{3}{5} v_3^2)^2 + \frac{5}{3}(v_{22} - v_2^2 - \frac{3}{5} v_3^2)^2 - \frac{5}{3} v_1^4 - \frac{5}{3} v_2^4 - \frac{6}{5} v_3^4$$
$$\quad - 2 v_1^2 v_3^2 - 2 v_2^2 v_3^2 + \frac{5}{3}|Dv|^4 + 2|Dv|^2 v_3^2 + (v_2^2 + v_3^2)(v_1^2 + v_3^2)] v_{113} v_{223}$$
$$\geq 4 v_2^2 v_1^2 v_{113} v_{223}. \tag{4.21}$$

If $v_1 v_2 \neq 0$, we have $v_{123}^2 \gtrsim v_{113} v_{223}$. Combining this with (4.19) and (4.20), we get (4.5). If $v_1 v_2 = 0$, then $F^{12} = -v_{21} + v_2 v_1 = 0$, putting this into (4.14), we have $F^{11} v_{113} + F^{22} v_{223} \sim 0$, from which we know

$$-v_{113} v_{223} \sim \frac{F^{11}}{F^{22}} v_{113}^2 \geq 0,$$

which also implies (4.5).

Case 2: $r_0 = 1$. In this case, $P = \sigma_2(D^2 v)$ and like case 1, there exists a positive constant $c_2 > 0$ such that $v_{11} \geq c_2 > 0$. Since $D^2 v$ is diagonal at x, we have

$$0 \leq v_{33} \leq v_{22} \leq v_{22} + v_{33} + \frac{v_{22} v_{33}}{v_{11}} = \frac{P}{v_{11}} \leq \frac{P}{c_2},$$

i.e.

$$v_{33} \sim v_{22} \sim 0 \sim P. \tag{4.22}$$

Differentiating P with respect to x_i,

$$v_{22i} + v_{33i} \sim P_i \sim 0. \tag{4.23}$$

Using the above fact and differentiating P with respect to x_i and x_j, we have

$$P_{ij} \sim -2v_{12i}v_{12j} - 2v_{13i}v_{13j} - 2v_{23i}v_{23j} + v_{11}(v_{22ij} + v_{33ij}) + (v_{22i}v_{33j} + v_{22j}v_{33i}). \tag{4.24}$$

Hence

$$F^{ij}P_{ij} \sim v_{11}F^{ij}(v_{22ij} + v_{33ij}) - 2F^{ij}(v_{12i}v_{12j} + 2v_{13i}v_{13j}) - 2F^{ij}(v_{23i}v_{23j} + v_{22i}v_{22j}). \tag{4.25}$$

To handle the first term, we differentiate the equation $F(Dv, D^2v) = \lambda$ with respect to x_2 and x_3 respectively. We then get

$$F^{p_l}v_{l2} + F^{ij}v_{ij2} = 0, \tag{4.26}$$

$$F^{p_l}v_{l22} + F^{ij,rs}v_{ij2}v_{rs2} + F^{ij}v_{ij22} \sim 0, \tag{4.27}$$

$$F^{p_l}v_{l3} + F^{ij}v_{ij3} = 0, \tag{4.28}$$

$$F^{p_l}v_{l33} + F^{ij,rs}v_{ij3}v_{rs3} + F^{ij}v_{ij33} \sim 0. \tag{4.29}$$

Adding (4.29) to (4.27) we get

$$F^{ij}(v_{22ij} + v_{33ij}) \sim -F^{ij,rs}(v_{ij2}v_{rs2} + v_{ij3}v_{rs3}). \tag{4.30}$$

This together with (4.25) gives

$$F^{ij}P_{ij} \sim -v_{11}F^{ij,rs}(v_{ij2}v_{rs2} + v_{ij3}v_{rs3}) - 2F^{ij}(v_{12i}v_{12j} + 2v_{13i}v_{13j})$$
$$- 2F^{ij}(v_{23i}v_{23j} + v_{22i}v_{22j}). \tag{4.31}$$

To proceed, we need the following lemma (see [4]):

Lemma 4.2. *Let $v \in C^{3,1}(\Omega)$ be a convex function. Then for any $O \subset\subset \Omega$, there exists a positive constant C depending only on dist $(O, \partial\Omega)$ and $\|v\|_{C^{3,1}(\Omega)}$ such that*

$$|v_{ij\alpha}| \le C(\sqrt{v_{ii}} + \sqrt{v_{jj}}) \tag{4.32}$$

for all $x \in O$ and $1 \le i, j, \alpha \le 3$.

Using this lemma, Cauchy inequality and (4.22), we get

$$v_{ijk}^2 \sim 0, \quad 2 \le i, j \le 3; 1 \le k \le 3. \tag{4.33}$$

On the other hand, from (4.26) and (4.28), we know

$$F^{ij}v_{ij2} \sim F^{ij}v_{ij3} \sim 0. \tag{4.34}$$

Putting (4.33) into (4.34), we get from Cauchy inequality that $(F^{11}v_{112})^2 \sim (F^{11}v_{113})^2 \sim 0$, i.e.

$$v_{112}^2 \sim v_{113}^2 \sim 0. \tag{4.35}$$

Putting (4.33) and (4.35) into (4.31), we get (4.5) from the Cauchy inequality. This completes the proof of the lemma. $\qquad\square$

5   Proof of Theorem 1.2

In this section, we use the constant rank theorem and the deformation technique to prove Theorem 1.2. We first prove the strict convexity of v when Ω is a ball. The proof is following the idea of McCuan [28].

Lemma 5.1 (Case in Ball). *Let $B_R(0)$ be a ball in R^3 with radius $R > 0$. Let $u \in C^\infty(B_R) \cap C^{1,1}(\overline{B_R})$ be the admissible eigenfunction for* (1.1) *in $B_R(0)$. Then $v = -\log(-u)$ is a strictly convex function in $B_R(0)$.*

Proof. By the uniqueness of solution for (1.1) up to a positive constant, we know the solution u is a radial function. We set

$$u(x) = \varphi(|x|) = \varphi(r) \quad for \ r = |x|. \tag{5.1}$$

where $r \in [0,R]$, then $\varphi < 0$ for $r \in [0,R)$ and $\varphi'(0) = \varphi(R) = 0$.

A straightforward calculation yields

$$u_{ij} = (\varphi'' r^{-2} - \varphi' r^{-3}) x_i x_j + \varphi' r^{-1} \delta_{ij}$$

and the three eigenvalues of $D^2 u$ are $\frac{\varphi'}{r}, \frac{\varphi'}{r}, \varphi''$. So, (1.1) can be rewritten as

$$\frac{5(\varphi')^2}{r^2} + \frac{6\varphi'\varphi''}{r} + (\varphi'')^2 = \lambda(-\varphi)^2 \quad in \ [0,R). \tag{5.2}$$

Since $v = -\log(-\varphi)$, we have

$$\varphi' = e^{-v} v', \quad \varphi'' = e^{-v}(v'' - (v')^2). \tag{5.3}$$

Putting this into (5.2), we get

$$5(v')^2 + 6[v'v'' - (v')^3]r + [(v')^4 + (v'')^2 - 2(v')^2 v'']r^2 = \lambda r^2 \quad in \ (0,R). \tag{5.4}$$

Since $\Delta u > 0$ in $B_R(0)$, i.e.

$$(\frac{\varphi'}{r}, \frac{\varphi'}{r}, \varphi'') \in \Gamma_1, \quad r \in [0,R) \tag{5.5}$$

and $\varphi'(0) = 0$, we have $\varphi''(0) > 0$. Thus

$$v''(0) = e^v \varphi''(0) + (v')^2(0) > 0. \tag{5.6}$$

If there exists some point $r_1 \in [0,R)$ such that $v''(r_1) \leq 0$, we let

$$r_0 = inf\{r_1 \in [0,R) | v''(r_1) \leq 0\}. \tag{5.7}$$

Then, $r_0 > 0$ by (5.6) and $v''(r_0) = 0$. Differentiating (5.4) and taking value at r_0, we get

$$-6(v')^3 + 6rv'v''' + 2r(v')^4 - 2r^2(v')^2 v''' = 2\lambda r. \tag{5.8}$$

Since $v''(r_0) = 0$, we know from (5.4) that

$$5(v')^2 - 6(v')^3 r + (v')^4 r^2 = \lambda r^2, \tag{5.9}$$

i.e.

$$2r(v')^4 = 12(v')^3 - \frac{10(v')^2}{r} + 2\lambda r. \tag{5.10}$$

Putting (5.10) into (5.8) to substitute $2r(v')^4$, we have

$$v'''(r_0) = \frac{-3(v')^2 + \frac{5v'}{r}}{3r - r^2 v'}. \tag{5.11}$$

This implies that $v'''(r_0)$ has the same sign as $3(rv')^3 - 14(rv')^2 + 15rv'$. We also notice that $\varphi''(r_0) = e^{-v}(v'' - (v')^2)(r_0) \leq 0$ for $v''(r_0) = 0$, hence by (5.5), we have $\varphi'(r_0) > 0$ which implies $v'(r_0) = e^v \varphi'(r_0) > 0$. So, $v'''(r_0)$ has the same sign as

$$3(rv')^2 - 14(rv') + 15, \tag{5.12}$$

which means that $v'''(r_0) > 0$ if $rv'(r_0) \in (-\infty, \frac{5}{3}) \cup (3, +\infty)$.

On the other hand, from (5.9), we know

$$5(rv')^2 - 6(rv')^3 + (rv')^4 = \lambda r^4 > 0. \tag{5.13}$$

Hence $rv'(r_0) \in (0,1) \cup (5,+\infty) \subset (-\infty, \frac{5}{3}) \cup (3,+\infty)$, $v'''(r_0) > 0$. However, this contradicts to the fact that $v''(r) > 0$ for $0 \leq r < r_0$ and $v''(r_0) = 0$. So we have $v''(r) > 0$ for $0 \leq r < R$, i.e. v is strictly convex in $B_R(0)$. □

Now, we state the following well-known boundary convexity lemma, see for example Caffarelli-Friedman [6](p. 450, Lemma 4.3) or Korevaar [22] (p. 610, Lemma 2.4).

Lemma 5.2. *Let $\Omega \subset R^3$ be smooth, bounded and strictly convex. Let $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ satisfies*

$$u < 0 \quad in \, \Omega, \qquad u = 0 \quad and \quad Du \cdot \nu > 0 \quad on \, \partial\Omega, \tag{5.14}$$

where ν is the exterior normal to $\partial\Omega$. Let

$$\Omega^\varepsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) > \varepsilon\} \tag{5.15}$$

and let $v = f(u)$. Then for small enough $\varepsilon > 0$ the function v is strictly convex in a boundary strip $\Omega - \Omega^\varepsilon$ if f satisfies

$$(i) \, f' > 0, \quad (ii) \, f'' > 0, \quad (iii) \lim_{u \to 0^-} \frac{f'}{f''} = 0. \tag{5.16}$$

Proof. The proof can be found in Korevaar [22]. We only point out that in Korevaar [22], u is assumed to be in $C^2(\overline{\Omega})$. But we can follow the calculations there to show that the similar result is true for our case when $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$. In fact, we know firstly that u has bounded second derivatives a.e. in $\overline{\Omega}$ since $u \in C^{1,1}(\overline{\Omega})$. Following the proof in [22], we then only need to get a lower bound for $\frac{\partial u}{\partial v}$ on $\partial\Omega$. However, this is known from Lemma 3.1. In fact, from (3.12) we know there exists $x_0 \in \Omega$, positive constants R and δ such that

$$u(x) \leq -2\delta, \quad x \in B_{2R}(x_0). \tag{5.17}$$

Then, Lemma 3.1 implies the desired result.   □

Now, we use the deformation technique and the constant rank theorem to prove Theorem 1.2 as in Korevaar-Lewis [23] and Ma-Xu [26, 27].

Proof. If Ω is the ball $B_R(0)$, by Lemma 5.1, we know $v = -\log(-u)$ is strictly convex in $B_R(0)$. For an arbitrary bounded strictly convex domain Ω, set

$$\Omega_t = (1-t)B_R(0) + t\Omega, \ 0 \leq t \leq 1.$$

Then, from the theory of convex bodies, we can deform $B_R(0)$ continuously into Ω by the family $\{\Omega_t\}$, $0 \leq t < 1$, of strictly convex domain in such a way that $\partial\Omega_t \to \partial\Omega_s$ as $t \to s$ in the sense of Hausdorff distance, whenever $0 \leq s \leq 1$. And the deformation also is chosen so that $\partial\Omega_t$, $0 \leq t < 1$, can be locally represented for some α, $0 < \alpha < 1$, by a function whose norm in the space $C^{2,\alpha}$ of functions with Holder continuous second derivatives depends only on δ, whenever $0 < t \leq \delta < 1$.

Suppose $u_t \in C^\infty(\Omega_t) \cap C^{1,1}(\overline{\Omega}_t)$ is the *admissible* solution of (1.1), $v_t := -\log(-u_t)$ and H_t is the corresponding hessian matrix of v_t. First, H_0 is positive definite, and from Lemma 5.2 we have H_δ is positive definite in an ε neighborhood of $\partial\Omega_\delta$. From Theorem 2.1 and 2.2, we know this bound depends only on the uniformly bounded geometry of Ω_t which depends on the geometry Ω and t. We conclude that if $v(.,s)$ is strictly convex for all $0 \leq s < t$, then $v(.,t)$ is convex.

So if for some δ, $0 < \delta < 1$, H_δ is positive semi-definite but not positive definite in Ω_δ, we say it is impossible by constant rank theorem and Lemma 5.2. We conclude H_δ is positive definite. Then $v = -\log(-u)$ is strictly convex in Ω.   □

Acknowledgments

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