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AN UNFITTED hp-INTERFACE PENALTY FINITE ELEMENT METHOD FOR ELLIPTIC INTERFACE PROBLEMS *

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Abstract

An hp version of interface penalty finite element method (hp-IPFEM) is proposed to solve the elliptic interface problems in two and three dimensions on unfitted meshes. Error estimates in broken H^1 norm, which are optimal with respect to h and suboptimal with respect to p by half an order of p, are derived. Both symmetric and non-symmetric IPFEM are considered. Error estimates in L^2 norm are proved by the duality argument. All the estimates are independent of the location of the interface relative to the meshes. Numerical examples are provided to illustrate the performance of the method. This paper is adapted from the work originally post on arXiv.com by the same authors (arXiv:1007.2893v1).

Mathematics subject classification: 65N12, 65N15, 65N30
Key words: Elliptic interface problems, Unfitted mesh, hp-IPFEM.

1. Introduction

In this paper we consider the following elliptic interface problem for u: Let $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ be a bounded and convex polygonal or polyhedral domain in $\mathbb{R}^d, d = 2$ or 3, where Ω_1 and Ω_2 are two subdomains of Ω and are separated by a C^2-smooth interface Γ,

$$\begin{cases} -\nabla \cdot \big(a(x)\nabla u\big) = f, & \text{in } \Omega_1 \cup \Omega_2, \\ [u] = g_D, \qquad \big[\big(a(x)\nabla u\big)\cdot \mathbf{n}\big] = g_N, & \text{on } \Gamma, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For definiteness, we define $\mathbf{n}$ as the unit outward normal to the boundary of Ω_1, which is strictly included in Ω, i.e. $\partial\Omega_1 = \Gamma$ and $\partial\Omega \cup \Gamma = \emptyset$ (see Fig. 1.1). In (1.1), we have used the notation $[v] = v|_{\Omega_1} - v|_{\Omega_2}$ for the jump of a function v across the interface Γ. The coefficient $a(x)$ is assumed to be C^1-smooth in each subdomain and bounded from below and above by some positive constants.

Due to the possible discontinuity of $a(x)$ across the interface Γ, the standard numerical methods, which are efficient for smooth solutions, usually lead to a loss of accuracy across the interface. One way to render a satisfactory approximation is to use interface-fitted/resolved grids (cf. [2–5]). In the interface-fitted approach, an element of the underlying mesh is required to intersect with the interface only through its boundaries. When the geometry is complex, this usually leads to a nontrivial interface meshing problem. One may face with severe challenges when the interface evolves with time or the mesh refinement is performed. In practice, it is readily to encounter the "inverted elements", i.e., the elements with negative directional

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areas/volumes (cf. [6]), near curved interfaces during interface-fitted local refinements, in particular, in the three dimensional case. From this point of view, it would be advantageous to use a method in which the interface are not necessarily aligned with the background mesh.

Many discretization techniques on unfitted meshes for the elliptic interface problems have been proposed and discussed in the literature, cf. for example [7–24]. The method we will consider below uses penalty terms in the vicinity of the interface to weakly enforce the transmission conditions between subdomains. Unfitted mesh methods involving penalty terms can be traced back to the *penalty finite element method* proposed by Babuška [7]. A. Hansbo and P. Hansbo [18] proposed an *unfitted finite element method* which can be viewed as a linear and consistent modification of Babuška's method. By introducing a geometry dependent average of flux at the interface, they derived a stable discretization and proved this *linear* finite element scheme is quasi-optimal in two dimensions. This approach has motivated many follow-up works, e.g., the unfitted finite element method [23,25–29], the Ghost penalty method [30], the cut finite element method [31], the unfitted discontinuous Galerkin methods [19,20]. Although significant progresses in the error analyses of some methods have been made, nevertheless, the majority of these method are built on piecewise linear discretizations. Some exceptions which claim high order approximation are found in [19, 23, 24, 26]. Massjung [19] proposed an *hp-unfitted discontinuous Galerkin method* for Problem (1.1) and show that, only for the two dimensional case, the method converges in broken H^1 norm at an optimal rate with respect to h and at a suboptimal rate with respect to p by a factor of p. In [26], an isoparametric finite element method with a high order geometrical approximation of level set domains is presented and discussed in detail. In [24] and a forthcoming work [23], various issues related to unfitted methods have been addressed, including the dependence of error estimates on the diffusion coefficients, the condition number of the discrete system, and the choice of stabilization parameters. We also refer to [32, 33] for different approaches to compute integrals on curved sub-elements and their curved boundaries.

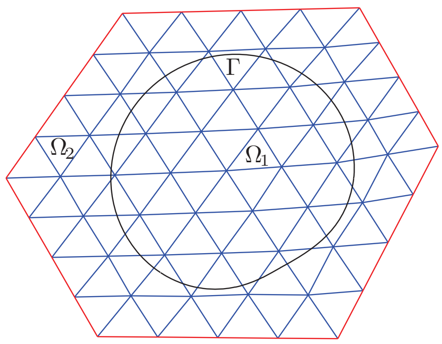

Fig. 1.1. A sample domain Ω and an unfitted mesh.

The goal of this paper is to propose a quasi-optimal *hp*-interface penalty finite element method (*hp*-IPFEM) for the interface problem (1.1) on unfitted meshes along with a rigorous error analysis in both two and three dimensions. In this method, the approximation space consists of functions whose restrictions to each $\Omega_i, i = 1, 2$ are merely the restrictions of standard continuous finite element functions, while the approximation functions are allowed to be dis-