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DERIVATIVE SAMPLING EXPANSIONS FOR THE LINEAR CANONICAL TRANSFORM: CONVERGENCE AND ERROR ANALYSIS *

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Abstract

In recent decades, the fractional Fourier transform as well as the linear canonical transform became very efficient tools in a variety of approximation and signal processing applications. There are many literatures on sampling expansions of interpolation type for bandlimited functions in the sense of these transforms. However, rigorous studies on convergence or error analysis are rare. It is our aim in this paper to establish sampling expansions of interpolation type for bandlimited functions and to investigate their convergence and error analysis. In particular, we introduce rigorous error estimates for the truncation error and both amplitude and jitter-time errors.

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1. Introduction

Throughout this article σ is a fixed positive number, $a, b, d \in \mathbb{R}$, $b \neq 0$, are arbitrary. We also denote by $S_n(t)$, $t \in \mathbb{R}$ to the sinc-function

$$S_n(t) := \operatorname{sinc}\left(\frac{\sigma}{b}(t-t_n)\right) = \begin{cases} \dfrac{\sin\left(\frac{\sigma}{b}(t-t_n)\right)}{\frac{\sigma}{b}(t-t_n)}, & t \neq t_n, \\ 1, & t = t_n, \end{cases} \tag{1.1}$$

where $t_n = n\pi b/\sigma$. There are several ways to define the linear canonical transform (LCT), depending on selecting a normalization constant, which affects the definition of its inverse. Without any loss of the generality of the results obtained in this paper, we choose the definition introduced in [16] and the associated sampling theorem. Thus we define the LCT to be

$$\mathcal{L}[f](x) = \frac{1}{\sqrt{2\pi i b}} \int_{-\infty}^{\infty} f(t) e^{\frac{i}{2}\left[\frac{a}{b}t^2 - \frac{1}{b}tx + \frac{d}{b}x^2\right]}\, dt. \tag{1.2}$$

The case $b = 0$ is not of any interest. We also assume that $ad \neq 0$.

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The LCT reduces to a fractional Fourier-type transform (FrFT) when take $a = d$. The FrFT normally attributed to Namais [20], see also [1, 18]. Nevertheless, traces of history of the FrFT points out to the works of Wiener [28] and Condon [9].

The space of bandlimited signals in the sense of the LCT is defined in literature, see e.g. [16], to be

$$B_\sigma^2 := \Big\{ f(t) \in L^2(\mathbb{R}) : \ \ \mathcal{L}[f](x) = 0, \ \forall \ |x| > \sigma \Big\}. \tag{1.3}$$

There are many papers that develop sampling theorems of Lagrange interpolation type for elements of B_σ^2, see e.g. [15, 16, 21, 24–27]. This extends the works on sampling theorems for the bandlimited signals in the sense of the FrFT, see e.g. [11, 22, 29], and the reference cited therein. Due to [16], if $f(t) \in B_\sigma^2$, then

$$f(t) = e^{-i(a/2b)t^2} \sum_{n=-\infty}^{\infty} e^{i(a/2b)t_n^2} f(t_n) S_n(t), \qquad t \in \mathbb{R}. \tag{1.4}$$

While nothing is said about the convergence of this theorem, it is understood that it holds in the pointwise sense. No other types of convergence is considered. Also papers on the error analysis associated with (1.4) are rare. To the best we know, the paper of the authors [4] is the only known one that investigates truncation, amplitude and time-jitter errors associated with (1.4).

As we have mentioned, the LCT and its associated sampling theorems play important role in signal and image processing. This involves, but not limited to, encryption, watermarking, classification, registration and medical imaging, see, e.g., [10, 14]. The LCT has also played a major role in the hamiltonian formulation of classical and quantum mechanics, see [5,19]. As the classical sampling theorem take a reasonable part in numerical mathematics, see, e.g., [7,17,23], the interpolation of B_σ^2-functions is expected to extend this approaches to a more general setting.

The aim of this paper is twofold. First of all, we derive a term-by-term derivative sampling theorem associated with (1.4). Absolute and uniform convergence are proved in the appropriate domains. This is done in the next section. Secondly, we investigate the error analysis associated with the derivatives of (1.4). This involves the derivations of rigorous estimates for the truncation error, amplitude error and the jitter-time error. This is accomplished in Sections 3-5 respectively. We conclude the paper with a separate section on numerical examples and graphical illustrations.

2. A Derivative Sampling Theorem

This section involves the derivation of the derivative sampling theorem for B_σ^2-bandlimited functions. The convergence properties are investigated in the complex plane. Hereafter r is a nonnegative integer. We first notice that since $f \in B_\sigma^2$, then Parseval's equality takes the form

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{\pi b}{\sigma} \sum_{n=-\infty}^{\infty} \left| f\left(\frac{n\pi b}{\sigma}\right)\right|^2. \tag{2.1}$$

This is because

$$\langle f, f\rangle = \sum_{n=-\infty}^{\infty} f\left(t_n\right) e^{i(a/2b)t_n^2} \sum_{m=-\infty}^{\infty} \overline{f}\left(t_m\right) e^{-i(a/2b)t_m^2} \int_{-\infty}^{\infty} S_n(t) S_m(t) dt,$$