

Collocation Methods for Cauchy Problems of Elliptic Operators via Conditional Stabilities

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Received 11 July 2018; Accepted (in revised version) 7 November 2018

Abstract. Ill-posed Cauchy problems for elliptic partial differential equations appear in many engineering fields. In this paper, we focus on stable reconstruction methods for this kind of inverse problems. Using kernels that reproduce Hilbert spaces $H^m(\Omega)$, numerical approximations to solutions of elliptic Cauchy problems are formulated as solutions of nonlinear least-squares problems with quadratic inequality constraints (LSQI). A convergence analysis with respect to noise levels and fill distances of data points is provided, from which a Tikhonov regularization strategy is obtained. A nonlinear algorithm using generalized singular value decomposition of matrices and Lagrange multipliers is proposed to solve the LSQI problem. Numerical experiments of two-dimensional cases verify our proved convergence results. By comparing with solutions of MFS and FEM under the discrete Tikhonov regularization by RKHS under same Cauchy data, we demonstrate that our method can reconstruct stable and high accuracy solutions for noisy Cauchy data.

AMS subject classifications: 65D15, 65N35, 65N21

Key words: Cauchy problems, meshfree, Kansa method, error analysis, LSQI problem, Tikhonov regularization.

1 Introduction

It is well known that Cauchy problems are ill-posed in the sense that their solutions do not continuously depend on data. However, Tikhonov [36] proposed that conditional stabilities of solutions for Cauchy problems can be constructed with an a priori bound to the exact solution. In [21], an interior stability for elliptic Cauchy problems was proved. A global stability was proved based on the Carleman estimate in [34] by Takeuchi and

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Yamamoto. There are many other interior and global conditional stability results for Cauchy problems, and for more detail, one can refer to [1, 5, 15].

Based on these conditional stabilities, efforts were made to look for stable numerical methods. The quasi-reversibility method [24] as regularization was proposed for solving Cauchy problems of Laplace equations by Klibanov in 1990 and convergence analysis for a discrete finite difference scheme was also given. In [3], a similar method with an adaptive regularization parameter selective strategy was proposed for inverse Cauchy problems. In [34], the discretized Tikhonov regularization was proposed by Takeuchi and Yamamoto. Their regularization was built on the theory of reproducing kernel Hilbert spaces (RKHS). A finite element scheme for Cauchy problems was used and convergence results of the method were also proved in the same paper. Other numerical methods with convergent analysis are found in the works [6, 19, 33].

Meshless methods are another popular numerical method for solving Cauchy problems. Generally speaking, these methods can be applied to complicated geometry and to solving high dimensional problems. The method of fundamental solution (MFS) with different regularization strategies was used to solve homogenous Cauchy problems in [16, 20, 40]. MFS combined with the method of particular solution (MPS) was used to solve nonhomogeneous cases by Li, Xiong, and Chen in [27, 38]. A meshless method called the general finite difference method (GFDM) was proposed by Fan in [9] to solve inverse Cauchy problems. These meshless approaches usually have good numerical performance. However, most, if not all, are ad hoc and do not have robust theoretical support.

Recently, some progress has been made in the theoretical aspects of meshless collocation methods for PDEs. The Kansa method, proposed by E. J. Kansa in 1990 [22, 23], is a typical meshless method used to solve partial differential equations (PDEs) by imposing strong form collocation conditions to PDEs. To overcome the singular problem of matrix systems by the Kansa method appearing in some cases [18], the overdetermined Kansa method was applied to solve PDEs in [29]. Partial convergence results of the overdetermined Kansa method were proved by Ling and Schaback in [30]. Recently, convergence theorems for overdetermined Kansa methods for elliptic PDEs were proved by Cheung, Ling, and Schaback in [7].

Motivated by these improvements, in this paper, we apply an overdetermined kernel-based collocation formulation to solve inverse Cauchy problems. In Section 2, we first introduce Cauchy problems considered in this paper and make some assumptions. We define discrete solutions for Cauchy problems with exact Cauchy data in some trial spaces of the symmetric positive definite kernel. The discrete solutions were defined as solutions of nonlinear optimization problems with quadratic inequality constraints. In the definition, the Tikhonov regularization strategy is used. Convergence results of discrete solutions with respect to data densities and noise levels are also proved based upon the scattered data approximation theory in RKHS [10, 37]. The value of the regularization parameter can also be fixed in the proof. After considering exact Cauchy data, we also define discrete solutions with noisy Cauchy data as solutions of nonlinear least-squares