Bound state solutions of the *s*-wave Klein-Gordon equation with position dependent mass for exponential potential

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Abstract. Bound state solutions of the *s*-wave Klein-Gordon equation with spatially dependent exponential-type mass for exponential-type scalar and vector potential are studied by using the Nikiforov-Uvarov method. The wave functions of the system are taken on the form of the Laguerre polynomials and the energy spectra of the system are discussed. In limit of constant mass, the wave functions and energy eigenvalues are in good agreement with the results previously.

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Key words: Klein-Gordon equation, Bound state solution, position dependent mass, exponential potential

1 Introduction

When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics. Taking the relativistic effect into account, one can apply the Klein-Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle. In fact, the problem of exact solutions of the Klein-Gordon equation for a number of special potential has also been a line of great interest in the recent years [1–7]. For example, some authors assumed that the scalar potential is equal to the vector potential and obtained the exact solutions of the klein-Gordon equation with some typical potential by using different methods. These investigations include the harmonic oscillator [8], the triaxial and axially deformed harmonic oscillators potential [9], Eckart potential [10, 11], Woods-Saxon potential [12], pseudoharmonic oscillator [13], ring-shaped Kratzer-type potential [14], ring-shaped non-spherical oscillator [15],

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double ring-shaped oscillator [16], Hartmann potential [17, 18], Rosen-Morse-type potential [19], generalized symmetrical double-well potential [20], Scarf-type potential [21], etc. These methods include the standard method, supersymmetry quantum mechanics [10], the Nikiforov-Uvarov (NU) method [12, 22–24], and others.

On the other hand, the concept of the position dependent mass in the quantum mechanical systems has also attracted a lot of attention and inspired intense research activities. They are indeed very useful and have been widely used in many different fields, such as semiconductor physics [25], quantum wells and quantum dots [26], He clusters [27],quantum liquids [28] and semicondector heterostructures [29], etc. In recent years, the solutions of the nonrelativistic wave equation with position dependent mass have been a line of great interest [30–34] but there are only few contributions that give the solution of the relativistic wave equation with position dependent mass in the quantum mechanics. Alhaidari [35] studied the exact solution of the Dirac equation with position dependent mass in the Coulomb field. Vakarchuk [36] investigated the Kepler problem in Dirac theory for a particle whose potential and mass are inversely proportional to the distance from the force center. Jia *et al.* investigated the approximately solution of the one-dimensional Dirac equations with spatially dependent mass for the generalized Hulthen potential [37]. Jia and Souza Dutra [38] considered position-dependent effective mass Dirac equations with PT and non-PT symmetric potential. In Ref. [39], Souza Dutra and Jia investigated the exact solution of the one-dimensional Klein-Gordon equation with spatially dependent mass for the inversely linear potential. Here we intend to study the one-dimensional Klein-Gordon equation for the exponential potential with an exponentially spatially dependent mass. We solve the equation by using the Nikiforov-Uvarov method [40] and discuss the limit of the constant-mass. The organization of this paper consists of three sections: In Section 2, we review the Nikiforov-Uvarov method briefly. Section 3 is devoted to the analytic bounded solutions of the Klein-Gordon equation for this quantum system by the NU method. Finally, the relevant results are discussed in Section 4.

2   Nikiforov-Uvarov method

The NU method is based on solving the second-order linear differential equation by reducing to a generalized equation of hypergeometric type.The NU method has been used to solve the Schrodinger,Dirac and Klein-Gordon wave equations for certain kind of potential [41]. In this method, the second-order differential equation can be written in the following form

$$\psi(s)'' + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. In order to find a particular solution to Eq. (1), we use the following transformed

$$\psi(s) = \phi(s)y(s). \tag{2}$$

It reduces the Eq. (1) to an equation of hypergeometric type

$$\sigma(s)y'' + \tau(s)y' + \lambda y = 0, \tag{3}$$

where $\phi(s)$ is defined as a logarithmic derivative

$$\phi'(s)/\phi(s) = \pi(s)/\sigma(s). \tag{4}$$

The other part $y(s)$ is the hypergeometric-type function whose polynomial solution are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left(\sigma^n(s)\rho(s) \right), \tag{5}$$

where B_n is a normalizing constant and weight function $\rho(s)$ must satisfy the condition

$$(\sigma\rho)' = \tau\rho. \tag{6}$$

The function π and the parameter λ required for this method are defined as follows

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma}, \tag{7}$$

$$\lambda = k + \pi'. \tag{8}$$

On the other hand, in order to find the value of k, the expression under the square root must be the square of the polynomial. Thus, a new eigenvalue equation for the second-order differential equation becomes

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \tag{9}$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tag{10}$$

and its derivative is negative. By comparison the Eq. (8) and Eq. (9), we obtain the energy eigenvalues.

3   Bound state of solutions in terms of the NU method

In the relativistic atomic units ($\hbar = c = 1$), for a spinless particle, the three-dimensional radial s-wave Klein-Gordon equation with position dependent mass is written as follows [39]

$$\left(\frac{d^2}{dr^2} + \left(E - V(r) \right)^2 - \left(m(r) + S(r) \right)^2 \right) u(r) = 0, \tag{11}$$

where the radial wave functions is $\phi(r) = u(r)/r$, and $V(r)$ and $S(r)$ are vector and scalar potentials respectively. From Eq. (11), we have

$$\frac{d^2u(r)}{dr^2} + \left(E^2 - m^2 - 2mS - 2EV + V^2 - S^2\right)u(r) = 0. \tag{12}$$

If we consider the case exponential-type scalar and vector potential case [41,42], the potentials can be written as

$$S(r) = -S_0 e^{-\alpha r}, \qquad V(r) = -V_0 e^{-\alpha r}, \tag{13}$$

where S_0, V_0 and α are constant. We take a specific form of the position dependent mass

$$m(r) = m_0(1 - qe^{-\alpha r}). \tag{14}$$

Substituting the Eqs. (13) and (14) into the Eq. (12), we obtains

$$\frac{d^2u(r)}{dr^2} + \left(-K_1 e^{-2\alpha r} + K_2 e^{-\alpha r} + E^2 - m_0^2\right)u(r) = 0, \tag{15}$$

where

$$K_1 = (S_0 + m_0 q)^2 - V_0^2, \qquad K_2 = 2(EV_0 + m_0^2 q + m_0 S_0). \tag{16}$$

Defining a new variable $z = e^{-\alpha r}$ and substituting it into Eq. (15), we obtain the following equation

$$\frac{d^2u(z)}{dz^2} + \frac{1}{z}\frac{du(z)}{dz} + \frac{-\gamma z^2 + \beta z - \epsilon^2}{z^2}u(z) = 0, \tag{17}$$

where

$$\epsilon^2 = \frac{m_0^2 - E^2}{\alpha^2}, \qquad \beta = \frac{K_2}{\alpha^2}, \qquad \gamma = \frac{K_1}{\alpha^2}. \tag{18}$$

To solve Eq. (17), we apply the NU method in the present case. By comparing Eq. (17) with Eq. (1), the following expressions are obtained as

$$\tilde{\tau} = 1, \qquad \sigma = z, \qquad \tilde{\sigma} = -\gamma z^2 + \beta z - \epsilon^2. \tag{19}$$

Substituting the above expressing into Eq. (7), we have the function π

$$\pi(z) = \pm\sqrt{\gamma z^2 + (k - \beta)z + \epsilon^2}. \tag{20}$$

According to the NU method, the expression in the square root must be the square of the polynomial. Then the solution of Eq. (20) gives two roots of k individually,

$$k_{1,2} = \beta \pm 2\sqrt{\gamma}\epsilon. \tag{21}$$

In view of that, we can find two possible function π for each k as

$$\pi = \begin{cases} \pm(\sqrt{\gamma}z + \epsilon), & \text{for } k_1 = \beta + 2\sqrt{\gamma}\epsilon, \\ \pm(\sqrt{\gamma}z - \epsilon), & \text{for } k_2 = \beta - 2\sqrt{\gamma}\epsilon, \end{cases} \tag{22}$$

where k is determined by the polynomial $\tau = \tilde{\tau} + 2\pi$ has a negative derivative. The most suitable form of $\pi(z)$ is selected as

$$\pi = -\sqrt{\gamma}z + \epsilon, \qquad \text{for } k_2 = \beta + 2\sqrt{\gamma}\epsilon. \tag{23}$$

Hence $\tau(z)$ and $\tau'(z)$ are obtained as follows

$$\tau(z) = -2\sqrt{\gamma}z + 2\epsilon + 1, \qquad \tau'(z) = -2\sqrt{\gamma} < 0. \tag{24}$$

According to Eq. (8) and Eq. (9), we have

$$\lambda = \beta - 2\sqrt{\gamma}\epsilon - \sqrt{\gamma}, \tag{25}$$

$$\lambda_n = 2n\sqrt{\gamma}. \tag{26}$$

Letting $\lambda = \lambda_n$, the relation of the values n and the constant ϵ can be obtained as

$$\epsilon = -\frac{(2n+1)\sqrt{\gamma} - \beta}{2\sqrt{\gamma}}. \tag{27}$$

Substituting the Eq. (18) into the Eq. (27), one obtains

$$E^2 = m_0^2 - \left(\frac{1}{2}\frac{K_2}{\sqrt{K_1}} - \alpha\left(n + \frac{1}{2}\right) \right)^2, \tag{28}$$

By using the Eq. (16), the exact energy eigenvalues of the Klein-Gordon equation for this system are derived as

$$E_n(q) = \frac{-B_n(q) \pm \sqrt{B_n^2(q) - 4(S_0 + m_0 q)^2 C_n(q)}}{2(S_0 + m_0 q)^2}, \tag{29}$$

where

$$B_n(q) = -2\alpha\left(n + \frac{1}{2}\right)V_0\sqrt{(S_0 + m_0 q)^2 - V_0^2} + 2V_0 m_0(S_0 + m_0 q), \tag{30}$$

$$c_n(q) = \alpha^2\left(n + \frac{1}{2}\right)^2\left((S_0 + m_0 q)^2 - V_0^2\right)$$
$$- 2\alpha\left(n + \frac{1}{2}\right)m_0(S_0 + m_0 q)\sqrt{(S_0 + m_0 q)^2 - V_0^2} + m_0^2 V_0^2. \tag{31}$$

Let us now find the corresponding eigenfunctions for this system. Using Eq. (4) and Eq. (6), we can find

$$\phi(z) = z^\epsilon e^{-\sqrt{\gamma}z}, \tag{32}$$

$$\rho(z) = z^{2\epsilon} e^{-2\sqrt{\gamma}z}. \tag{33}$$

Substituting Eq. (33) into Eq. (5), the polynomial $y_n(z)$ can be found as follows

$$y_n(z) = B_n z^{-2\epsilon} e^{2\sqrt{\gamma}z} \frac{d^n}{dz^n} \left(z^{n+2\epsilon} e^{-2\sqrt{\gamma}z} \right). \tag{34}$$

By using $u(z) = \phi(z) y(z)$, the solution of the Eq. (12) can be written as

$$u(z) = B_n z^{-\epsilon} e^{\sqrt{\gamma}z} \frac{d^n}{dz^n} \left(z^{n+2\epsilon} e^{-2\sqrt{\gamma}z} \right), \tag{35}$$

and it can be written in terms of the generalized Laguerre polynomials $L_n^m(z)$

$$u(z) = N_n z^{\epsilon} e^{-\sqrt{\gamma}z} L_n^{2\epsilon} (2\sqrt{\gamma}z). \tag{36}$$

In the end, the total radial wave function of the system is shown below

$$\phi_n(r) = N_n \frac{1}{r} \exp(-\alpha\epsilon r) \exp(-\sqrt{\gamma} e^{-\alpha r}) L_n^{2\epsilon} (2\sqrt{\gamma} e^{-\alpha r}), \tag{37}$$

where N_n is normalization constant.

Now that we have obtained the energy eigenvalues of the radial s-wave Klein-Gordon equation with position dependent mass for the exponential potential, we will start to discuss some particular cases.

(i) For constant mass case $q = 0$, the energy spectrum (29) can become as

$$E_n(0) = \frac{-B_n(0) \pm \sqrt{B_0^2(0) - 4S_0^2 C_n(0)}}{2S_0^2}, \tag{38}$$

where

$$B_n(0) = -2\alpha \left(n + \frac{1}{2} \right) V_0 \sqrt{S_0^2 - V_0^2} + 2m_0 S_0 V_0, \tag{39}$$

$$c_n(0) = \alpha^2 \left(n + \frac{1}{2} \right)^2 (S_0^2 - V_0^2) - 2\alpha \left(n + \frac{1}{2} \right) m_0 S_0 \sqrt{S_0^2 - V_0^2} + m_0^2 V_0^2. \tag{40}$$

This result is the same as in Refs. [41,42]. In this circumstance, if vector potential is stronger than the scalar potential ($V_0 > S_0$), there is no a bound state for Klein-Gordon particle.

(ii) For $q \neq 0$, the energy spectrum is determined by the Eq. (29). If we use S_0 instead of $S_0 + m_0 q$ in Eq. (29), then the Eq. (29) is the same as Eq. (38). This shows that the mass $m_0 q$ only play an additional scalar potential role. In the case of that scalar potential is equal to vector potential ($S_0 = V_0$), the parameter K_1 is always positive values and there are bounded solutions for particles. If we consider the pure vector

potential ($S_0=0, V\neq 0$), there are always a bound states as long as $m_0q>V_0$. When we consider the pure scalar potential ($S_0\neq 0, V_0=0$), the energy levels given by

$$E_n(q)=\pm\sqrt{\alpha(n+1/2)\big(2m_0-\alpha(n+1/2)\big)} \tag{41}$$

have nothing to do with the parameter S_0 and q and this is in good agreement with the result of the paper [42]. Finally, we consider the free particle case ($S_0=V_0=0$) and take $\alpha=m_0=1$, the energy spectrum is given by

$$E_n(q)=\pm\sqrt{\left(n+\frac{1}{2}\right)\left(\frac{3}{2}-n\right)}, \tag{42}$$

so that there are only two lowest bound state $n=0$ and $n=1$.

4 Conclusions

In conclusion, we have studied the bound state solutions of the s-wave Klein-Gordon with position dependent mass for exponential-type scalar and vector potentials by using the Nikiforov-Uvarov method. The wave functions are given by the generalized Laguerre polynomials and the energy of this system are given the Eq. (29). We introduce an exponential-type variable mass that is the same with add scalar potential. In limit of constant-mass $q=0$, the energy equation is consistent with the results previously.

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