Journal of Computational Mathematics
Vol.37, No.4, 2019, 475–487.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1805-m2017-0184

THE STRUCTURE-PRESERVING METHODS FOR THE DEGASPERIS-PROCESI EQUATION*

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Abstract

This paper gives several structure-preserving schemes for the Degasperis-Procesi equation which has bi-Hamiltonian structures consisted of both complex and non-local Hamiltonian differential operators. For this sake, few structure-preserving schemes have been proposed so far. In our work, based on one of the bi-Hamiltonian structures, a symplectic scheme and two new energy-preserving schemes are constructed. The symplecticity comes straightly from the application of the implicit midpoint method on the semi-discrete system which is proved to remain Hamiltonian, while the energy conservation is derived by the combination of the averaged vector field method of second and fourth order, respectively. Some numerical results are presented to show that the three schemes do have the advantages in numerical stability, accuracy in long time computing and ability to preserve the invariants of the DP equation.

Mathematics subject classification: 65D15, 65L05, 65L70, 65P10
Key words: Degasperis-Procesi equation, bi-Hamiltonian structure, Structure-preserving scheme, Fourier pseudospectral method.

1. Introduction

In this paper, our interest is to study the Degasperis-Procesi equation [1] in the context of integrable systems, which belongs to a family of third-order dispersive nonlinear equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}. \tag{1.1}$$

The family of PDEs (1.1) includes two important equations: the Camassa-Holm (CH) equation (when $b = 2$) [2, 3] and the Degasperis-Procesi (DP) equation (when $b = 3$) [1]. Both the CH and DP equations can be viewed as the model of the shallow water and admit peakon solutions with a superposition of the form $u(x,t) = ce^{-|x-ct|}, c > 0$. Despite the only difference

* Received July 23, 2017 / Revised version received September 11, 2017 / Accepted May 21, 2018 / Published online February 1, 2019 /

of the associated parameter b, the DP equation is totally different from the CH equation. The DP equation has not only the peakon solution but also the shock peakon solution such as $u(x,t) = -\frac{1}{t+c}\text{sign}(x)e^{-|x|}, c > 0$ which produces the difficulty to capture the shock wave numerically. As a consequence, compared with the CH equation, there are only few numerical methods for the DP equation, including the operator splitting methods [4,5], the conservative finite difference methods [6,7], the particle method [8], the Fourier spectral method [9], the local discontinuous Galerkin method [10] as well as the direct discontinuous Galerkin method [11].

When designing a numerical algorithm, a natural ideal is to construct a numerical integrator which can carry as much as possible the intrinsic properties of the original systems [12]. In the derivation of the completely integrability [1], the bi-Hamiltonian structures of the DP equation are proposed in the forms

$$m_t = B_0\frac{\delta\mathcal{H}_{-1}}{\delta m} = B_1\frac{\delta\mathcal{H}_0}{\delta m}, \tag{1.2}$$

where $m = u - u_{xx}$, B_0, B_1 represent two skew-adjoint operators as

$$B_0 = \partial_x(1-\partial_x^2)(4-\partial_x^2), \quad B_1 = m^{\frac{2}{3}}\partial_x m^{\frac{1}{3}}\cdot(\partial_x - \partial_x^3)^{-1}m^{\frac{1}{3}}\partial_x m^{\frac{2}{3}}, \tag{1.3}$$

where $\partial_x m\cdot v := (mv)_x$. The associated two invariants read

$$\mathcal{H}_{-1} = -\frac{1}{6}\int u^3 dx, \quad \mathcal{H}_0 = -\frac{9}{2}\int u - u_{xx}dx. \tag{1.4}$$

Therefore, it is straightly to consider symplectic integrators for the DP equation based on these bi-Hamiltonian structures to preserve the symplecticity of the solution. Theoretical analysis and numerical experiments have demonstrated that the symplectic integrators are superior in long-time stability over conventional numerical methods with respect to a wide class of Hamiltonian systems. For more details, please refer to [12–14] and references therein. In addition to the symplecticity of the Hamiltonian system (1.2), the preservation of the invariant $\mathcal{H}_{-1}$ or $\mathcal{H}_0$ is also an important aspects of structure-preserving algorithms, which is usually referred as the energy-preserving method. The maintenance of system energy can also guarantee the stability in numerical simulations over a long period.

The general framework to construct symplectic or energy-preserving methods for infinite-dimensional Hamiltonian systems is the method of lines, which first discretizes the spatial derivatives and then results a system of Hamiltonian ODEs. It is not obvious for the DP equation due to the complex bi-Hamiltonian structures and the non-local operators. In this paper, we adopt the high-precision Fourier pseudospectral method for the spatial discretization because under the periodic boundary condition the corresponding spectral differential matrices can automatically maintain the symmetry of the discrete operator as that in the continuous case. Moreover, the extra computational cost caused by the density of the differential matrices can be easily resolved by the fast Fourier transform (FFT) [15]. Once the semi-discrete Hamiltonian ODEs are achieved, the symplectic or energy-preserving schemes can be systematically constructed in conjunction with any temporal symplectic integrators [13] or the popular averaged vector field (AVF) method [16].

The rest of this paper is organized as follows. In Section 2, we briefly introduce the symplectic integrator. In Section 3, a symplectic scheme and two energy-preserving schemes are constructed. Numerical experiments are presented to verify the superior behavior of the proposed schemes in Section 4.