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Global Existence and Blow-Up in a $p(x)$-Laplace Equation with Dirichlet Boundary Conditions

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Abstract. This paper is devoted to a $p(x)$-Laplace equation with Dirichlet boundary. We obtain the existence of global solution to the problem by employing the method of potential wells. On the other hand, we show that the solution will blow up in finite time with $u_0 \not\equiv 0$ and nonpositive initial energy functional $J(u_0)$. By defining a positive function $F(t)$ and using the method of concavity we find an upper bound for the blow-up time.

AMS subject classifications: 35J25, 35J65

Key words: $p(x)$-Laplace equation, global weak solution, finite time blow-up, upper bounds.

1 Introduction

In this paper, we consider the following $p(x)$-Laplace equation:

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q(x)-2}u, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$, $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$, $u_0 \not\equiv 0$, $p(\cdot)$, $q(\cdot) \in C(\overline{\Omega})$ and satisfy:

$$1 < p_- \le p(x) \le p_+ < q_- \le q(x) \le q_+ < +\infty.$$

We denote by $p_- = ess\inf_{x\in\Omega} p(x)$ and $p_+ = ess\sup_{x\in\Omega} p(x)$.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is an interesting topic (see [20, 21]). It arises from the nonlinear elasticity theory, electro-rheological fluids, we refer to [15] and [16]. This fluids have the

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interesting property that their viscosity depends on the electric field in the fluid. For a general account of the underlying physics and the mathematical theory, we refer to [14] and [15]. Liu and Zhao [2] studied an initial boundary value problem of semilinear hyperbolic equations and gave a threshold result of global existence and nonexistence of solutions. Here, they generalized the so-called potential well method to study the problem. This method was established by Payne and Sattinger in [1] and was a powerful technique in treating many problems (see, e.g., [3-5, 10]). Alaoui and Khenous in [6] studied the following problem:

$$\begin{cases} u_t-\mathrm{div}(|\nabla u|^{m(x)-2}\nabla u)=|u|^{p(x)-2}u+f, & Q=\Omega\times(0,T),\\ u=0, & \partial Q=\partial\Omega\times[0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. The author shows that any solution with nontrivial initial datum blows up in finite time when $f\equiv 0$. The following problem:

$$\begin{cases} u_t=\mathrm{div}(|\nabla u|^{p-2}\nabla u)+|u|^q-\frac{1}{|\Omega|}\int_\Omega|u|^q dx, & (x,t)\in\Omega\times(0,T),\\ \frac{\partial u}{\partial n}=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

was studied in [7], where global existence results (for any initial energy) were obtained with $q\leq 1$. Moreover, the solution must blow-up in finite time with $q>1$ and non-positive initial energy associated. However, they didn't give upper or lower bound for the blow-up time. Pao [11] discussed the following problem:

$$\begin{cases} u_t-Du_{xx}+c_0u=0,\\ -u_x(t,0)=\sigma u^{1+\gamma}(t,0), \qquad u_x(t,l)=\sigma u^{1+\gamma}(t,l),\\ u(0,x)=u_0(x), \end{cases} \tag{1.4}$$

where $x\in[0,l], t\in[0,T]$, D,c_0,σ, and γ are positive constants. By constructing a concave function, the author obtained the blowing-up property of the solution.

Motivated by the above work, we intend to study the global existence and the blow-up phenomena for the problem (1.1). By applying the method of potential wells and the method of concavity, we obtained the existence of global weak solution, the property of blow-up at finite time and upper bound for finite time blow-up. This paper is organized as follows. In Section 2, we give some preparation knowledge which will be used later. In Section 3, we discuss the existence of global weak solution. Section 4 is devoted to discussing the finite time blow-up and finding an upper bound for the blow-up time.

2 Preparation of manuscript

In order to deal with the $p(x)$-Laplacian problem, in this section, we give some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ ([18] and [19]) and some definitions.