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Variable Besov Spaces: Continuous Version

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Abstract. We introduce Besov spaces with variable smoothness and integrability by using the continuous version of Calderón reproducing formula. We show that our space is well-defined, i.e., independent of the choice of basis functions. We characterize these function spaces by so-called Peetre maximal functions and we obtain the Sobolev embeddings for these function spaces. We use these results to prove the atomic decomposition for these spaces.

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1 Introduction

Function spaces play an important role in harmonic analysis, in the theory of differential equations and in almost every other field of applied mathematics. Some of these function spaces are Besov spaces. The theory of these spaces has been developed in detail in [47] and [48] (and continued and extended in the more recent monographs [49] and [50]), but has a longer history already including many contributors; we do not want to discuss this here. For general literature on function spaces we refer to [1, 6, 22-23, 26, 41-43, 47-51] and references therein.

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [44] and PDE's with non-standard growth conditions. In these applications the Lebesgue $L^{p(\cdot)}$ and Sobolev spaces $W^k_{p(\cdot)}$ with variable integrability seem to appear in a natural way.

The systematic study of variable exponent Lebesgue spaces $L^{p(\cdot)}$ and Sobolev spaces $W^k_{p(\cdot)}$ started in [35]. Almeida and Samko [5] and Gurka, Harjulehto and Nekvinda [25] introduced variable exponent Bessel potential spaces $\mathcal{L}^{\alpha,p(\cdot)}$ with constant $\alpha \in \mathbb{R}$.

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The concept of function spaces with variable smoothness and the concept of variable integrability were first merged by Diening, Hästö and Roudenko in [15]. They defined Triebel-Lizorkin spaces $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and proved a discretization by the so-called φ-transform. Also atomic and molecular decomposition of these function spaces are obtained and used to derive trace results. When α, p, q are constants they coincide with the usual function spaces $F^{\alpha}_{p,q}$.

Besov spaces of variable smoothness and integrability initially appeared in the paper of Almeida and Hästö [4], while Xu [56] introduced Besov and Triebel-Lizorkin spaces with variable integrability. Several basic properties were shown, such as the Fourier analytical characterization, Sobolev embeddings and the characterization in terms of Nikolskij representations involving sequences of entire analytic functions. Later, the present author characterized these spaces by local means and established the atomic characterization, see [18]. After that, Kempka and Vybíral [34] characterized these spaces by ball means of differences and also by local means, see [38], [29] and [39] for the duality and complex interpolation of these function spaces. The trace property of variable Besov and Triebel-Lizorkin spaces was obtained in [36] and [21], see also [40]. These type of function spaces are defined via the discrete Fourier analytic tool. For general literature on function spaces of variable smoothness and integrability we refer to [2-4, 9-20, 24, 31-33, 52-63].

Based on continuous characterizations of Besov spaces, we introduce a new family of function spaces of variable smoothness and integrability. In this paper we present some properties of these function spaces.

In order to do these, this paper is divided into five Section. First we give some preliminaries, where we fix some notation and recall some basic facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. We then define the Besov spaces $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. We prove a useful characterization of these spaces based on the so-called local means. The theorem on local means that is proved for Besov spaces of variable smoothness and integrability is highly technical and its proof is required new techniques and ideas. Moreover we present the relation between the discrete variable Besov spaces and the continuous one. Section 3 is devoted to the Sobolev embeddings properties of these spaces. Using the results from Sections 3 and 4, we prove in Section 5 the atomic decomposition for $\boldsymbol{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

2 Preliminaries

As usual, we denote by $\mathbb{N}_0$ the set of all non-negative integers. The notation $f \lesssim g$ means that $f \le cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means that $f \lesssim g \lesssim f$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the Lebesgue measure of E and χ_E