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Global Existence and Blow-Up in a $p(x)$-Laplace Equation with Dirichlet Boundary Conditions

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Abstract. This paper is devoted to a $p(x)$-Laplace equation with Dirichlet boundary. We obtain the existence of global solution to the problem by employing the method of potential wells. On the other hand, we show that the solution will blow up in finite time with $u_0 \not\equiv 0$ and nonpositive initial energy functional $J(u_0)$. By defining a positive function $F(t)$ and using the method of concavity we find an upper bound for the blow-up time.

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Key words: $p(x)$-Laplace equation, global weak solution, finite time blow-up, upper bounds.

1 Introduction

In this paper, we consider the following $p(x)$-Laplace equation:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q(x)-2}u, & (x,t)\in\Omega\times(0,T), \\ u=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$, $u_0\in L^\infty(\Omega)\cap W_0^{1,p(x)}(\Omega)$, $u_0\not\equiv 0$, $p(\cdot)$, $q(\cdot)\in C(\overline{\Omega})$ and satisfy:

$$1<p_-\le p(x)\le p_+<q_-\le q(x)\le q_+<+\infty.$$

We denote by $p_- = ess\inf_{x\in\Omega} p(x)$ and $p_+ = ess\sup_{x\in\Omega} p(x)$.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is an interesting topic (see [20, 21]). It arises from the nonlinear e-lasticity theory, electro-rheological fluids, we refer to [15] and [16]. This fluids have the

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interesting property that their viscosity depends on the electric field in the fluid. For a general account of the underlying physics and the mathematical theory, we refer to [14] and [15]. Liu and Zhao [2] studied an initial boundary value problem of semilinear hyperbolic equations and gave a threshold result of global existence and nonexistence of solutions. Here, they generalized the so-called potential well method to study the problem. This method was established by Payne and Sattinger in [1] and was a powerful technique in treating many problems (see, e.g., [3-5, 10]). Alaoui and Khenous in [6] studied the following problem:

$$\begin{cases} u_t-\mathrm{div}(|\nabla u|^{m(x)-2}\nabla u)=|u|^{p(x)-2}u+f, & Q=\Omega\times(0,T),\\ u=0, & \partial Q=\partial\Omega\times[0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. The author shows that any solution with nontrivial initial datum blows up in finite time when $f\equiv 0$. The following problem:

$$\begin{cases} u_t=\mathrm{div}(|\nabla u|^{p-2}\nabla u)+|u|^q-\frac{1}{|\Omega|}\int_\Omega |u|^q dx, & (x,t)\in\Omega\times(0,T),\\ \frac{\partial u}{\partial n}=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

was studied in [7], where global existence results (for any initial energy) were obtained with $q\le 1$. Moreover, the solution must blow-up in finite time with $q>1$ and non-positive initial energy associated. However, they didn't give upper or lower bound for the blow-up time. Pao [11] discussed the following problem:

$$\begin{cases} u_t-Du_{xx}+c_0u=0,\\ -u_x(t,0)=\sigma u^{1+\gamma}(t,0), \qquad u_x(t,l)=\sigma u^{1+\gamma}(t,l),\\ u(0,x)=u_0(x), \end{cases} \tag{1.4}$$

where $x\in[0,l],t\in[0,T]$, D,c_0,σ, and γ are positive constants. By constructing a concave function, the author obtained the blowing-up property of the solution.

Motivated by the above work, we intend to study the global existence and the blow-up phenomena for the problem (1.1). By applying the method of potential wells and the method of concavity, we obtained the existence of global weak solution, the property of blow-up at finite time and upper bound for finite time blow-up. This paper is organized as follows. In Section 2, we give some preparation knowledge which will be used later. In Section 3, we discuss the existence of global weak solution. Section 4 is devoted to discussing the finite time blow-up and finding an upper bound for the blow-up time.

2 Preparation of manuscript

In order to deal with the $p(x)$-Laplacian problem, in this section, we give some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ ([18] and [19]) and some definitions.

Let

$$L^{p(x)}(\Omega)=\left\{u|u \text{ is measurable on } \Omega,\int_\Omega |u|^{p(x)}dx<\infty\right\}.$$

Let us introduce a norm on $L^{p(x)}(\Omega)$ by

$$\|u\|_{p(.),\Omega}=\|u\|_{L^{p(x)},\Omega}=\inf\left\{\lambda>0|\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx\le 1\right\},$$

and $(L^{p(x)}(\Omega),\|.\|_{p(.),\Omega})$ becomes a banach space, which we call a variable exponent Lebesgue space. The variable-exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denoted by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega)||\nabla u|\in L^{p(x)}(\Omega)\right\},$$

and can be equipped with the norm

$$\|u\|_{1,p(.)}=\|u\|_{p(.)}+\|\nabla u\|_{p(.)},\quad \forall u\in W^{1,p(x)}(\Omega).$$

The following conditions may be required in our results:

(P1) $\Omega\subset \mathbf{R}^N$ is an open bounded domain with the Lipschitz-continuous boundary;

(P2) $p(\cdot),q(\cdot)\in C(\overline{\Omega}):\Omega\to[1,\infty)$;

(P3) $\forall a.e.x\in\Omega,\ p(x)\in[p_-,p_+],\ q(x)\in[q_-,q_+]\subset(1,+\infty)$.

Lemma 2.1. *Let conditions (P2) and (P3) be fulfilled. Then for every $u\in L^{p(x)}(\Omega)$,*

$$\min\left\{\|u\|^{p_-}_{p(.),\Omega},\ \|u\|^{p_+}_{p(.),\Omega}\right\}\le\int_\Omega |u|^{p(x)}dx\le\max\left\{\|u\|^{p_-}_{p(.),\Omega},\ \|u\|^{p_+}_{p(.),\Omega}\right\}. \tag{2.1}$$

For the convenience of the following relate, we note (2.1) *as*

$$\|u\|^{p_1}_{p(.),\Omega}\le\int_\Omega |u|^{p(x)}dx\le\|u\|^{p_2}_{p(.),\Omega} \tag{2.2}$$

where p_1,p_2 equal to p_- or p_+.

Lemma 2.2 (The Poincaré inequality)**.** *Let Ω and $p(x)$ satisfy conditions (P1), (P2) and (P3), then there exists a finite constant $C>0$ such that for every $u\in W^{1,p(x)}_0(\Omega)$,*

$$\|u\|_{p(.),\Omega}\le C\|\nabla u\|_{p(.),\Omega}. \tag{2.3}$$

Lemma 2.3 (Embedding Theorem in Sobolev space with variable exponents)**.** *Let Ω and $p(x)$ satisfy conditions (P1), (P2) and (P3). If $q(x)\in C(\overline{\Omega})$ and $q(x)<p_*(x)$ in $\overline{\Omega}$, then for every $u\in W^{1,p(x)}_0(\Omega)$,*

$$\|u\|_{q(.),\Omega}\le C_*\|\nabla u\|_{p(.),\Omega}, \tag{2.4}$$

with a constant C_* depending on p_-,p_+ and N, the properties of $\partial\Omega$ and the modulus of continuity of $p(x)$. The embedding $W^{1,p(x)}_0(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.

Lemma 2.4. *Let Ω and $p(x)$ satisfy conditions (p1) and (p2) with $q=const\geq 1$. If $q\leq p(x)$ a.e. in Ω, then*

$$\|u\|_{q,\Omega}\leq C\|u\|_{p(.),\Omega} \text{ with the constant } C=(1+|\Omega|)^{\frac{1}{q}}. \tag{2.5}$$

Definition 2.1. *(Weak solution). $u=u(x,t)$ is called a weak solution of problem* (1.1) *on $\Omega\times(0,T)$, if $u\in L^\infty(0,T;W_0^{1,p(x)}(\Omega))$ with $u_t\in L^\infty(0,T;L^2(\Omega))$ and satisfies the problem* (1.1) *in the distribution sense, i.e.*

$$\int_\Omega u_t v dx+\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx=\int_\Omega |u|^{q(x)-2}uvdx, \quad \forall v\in W_0^{1,p(x)}. \tag{2.6}$$

Definition 2.2. *(Maximal existence time). Let $u(x,t)$ be a weak solution of* (1.1)*. We define the maximal existence time T of $u(x,t)$ as follows:*

(i) if $u(x,t)$ exists for all $0\leq t<\infty$, then $T=+\infty$;

(ii) if there exists a $t_0\in(0,+\infty)$ such that $u(x,t)$ exists for $0\leq t<t_0$, but does not at $t=t_0$, then $T=t_0$.

Definition 2.3. *(finite time blow-up). Let $u(x,t)$ be a weak solution of* (1.1)*. We call $u(x,t)$ blows up in finite time if the maximal existence time T is finite and*

$$\lim_{t\to T^-}\|u(\cdot,t)\|_2=+\infty. \tag{2.7}$$

Next, we define the functionals on $W_0^{1,p(x)}$ as follows:

$$J(u)=\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx,$$
$$I(u)=\int_\Omega |\nabla u|^{p(x)}dx-\int_\Omega |u|^{q(x)}dx.$$

3 Global existence

In this section, we want to discuss the global existence of the solution by applying the method of potential wells. Assume $p(x),q(x)$ satisfy conditions (P2), (P3) and $1<p_-\leq p(x)\leq p_+<q_-\leq q(x)\leq q_+<p_*(x)$, with

$$p_*(x)=\begin{cases} \frac{Np(x)}{(N-p(x))_+}, & \text{if } p_+<N. \\ +\infty, & \text{if } p_+\geq N. \end{cases}$$

The main results about global existence of the solution are as follows.

Theorem 3.1. *If $u_0\in W_0^{1,p(x)}(\Omega)$, $0<J(u_0)<d$ and $I(u_0)>0$, then the problem* (1.1) *has a global weak solution $u\in L^\infty(0,+\infty;W_0^{1,p(x)}(\Omega))$ with $u_t\in L^\infty(0,+\infty;L^2(\Omega))$. Moreover $u\in W$.*

Theorem 3.2. *If $u_0\in W_0^{1,p(x)}(\Omega)$, $J(u_0)=d$ and $I(u_0)\geq 0$, then the problem* (1.1) *admits a global weak solution $u\in L^\infty(0,+\infty;W_0^{1,p(x)}(\Omega))$ with $u_t\in L^\infty(0,+\infty;L^2(\Omega))$. Moreover $u\in\overline{W}$ for $t\in(0,\infty)$.*

Define

$$N=\Big\{u\in W_0^{1,p(x)}(\Omega)|I(u)=0,\ \int_\Omega|\nabla u|^{p(x)}dx\neq 0\Big\},$$
$$W=\Big\{u\in W_0^{1,p(x)}(\Omega)|I(u)>0,\ 0<J(u)<d\Big\},$$
$$d=\inf\Big\{\sup_{\lambda\geq 0}J(\lambda u)|u\in W_0^{1,p(x)},\ \int_\Omega|\nabla u|^{p(x)}dx\neq 0\Big\}.$$

For $\delta>0$, we also introduce

$$I_\delta(u)=\delta\int_\Omega|\nabla u|^{p(x)}dx-\int_\Omega|u|^{q(x)}dx.$$

Accordingly, we define

$$N_\delta=\Big\{u\in W_0^{1,p(x)}(\Omega)|I_\delta(u)=0,\ \int_\Omega|\nabla u|^{p(x)}dx\neq 0\Big\},$$
$$W_\delta=\Big\{u\in W_0^{1,p(x)}(\Omega)|I_\delta(u)>0,\ 0<J(u)<d(\delta)\Big\},$$
$$d(\delta)=\inf\Big\{\sup_{\lambda\geq 0}J_\delta(\lambda u)|u\in W_0^{1,p(x)},\ \int_\Omega|\nabla u|^{p(x)}dx\neq 0\Big\}.$$

Lemma 3.1. *For any given $u\in W_0^{1,p(x)}(\Omega)$ and $\|\nabla u\|_{p(.)}\neq 0$, let $g(\lambda)=J(\lambda u)$. Then*

(i) $\lim_{\lambda\to 0}J(\lambda u)=0$, $\lim_{\lambda\to+\infty}J(\lambda u)=-\infty$;

(ii) there must exists a unique $\lambda=\lambda^*$ such that $g'(\lambda^*)=0$ and

$$I(\lambda u)=\lambda g'(\lambda)=\lambda\frac{dJ(\lambda u)}{d\lambda}\begin{cases}>0, & 0<\lambda<\lambda^*,\\ =0, & \lambda=\lambda^*,\\ <0, & \lambda^*<\lambda<+\infty.\end{cases}$$

Proof. (i) Note that

$$g(\lambda)=J(\lambda u)=\int_\Omega\frac{\lambda^{p(x)}}{p(x)}|\nabla u|^{p(x)}dx-\int_\Omega\frac{\lambda^{q(x)}}{q(x)}|u|^{q(x)}dx.$$

Since $\|\nabla u\|_{p(.)}\neq 0$ and the property of $p(x)$ and $q(x)$, we obtain (i).
(ii) Observe that

$$\begin{aligned}g'(\lambda)&=\int_\Omega\lambda^{p(x)-1}|\nabla u|^{p(x)}dx-\int_\Omega\lambda^{q(x)-1}|u|^{q(x)}dx\\&=\lambda^{\widetilde{p}-1}\int_\Omega|\nabla u|^{p(x)}dx-\lambda^{\widetilde{q}-1}\int_\Omega|u|^{q(x)}dx.\end{aligned}$$

So, there exists a unique

$$\lambda=\lambda^*=\left(\frac{\int_\Omega|\nabla u|^{p(x)}dx}{\int_\Omega|u|^{q(x)}dx}\right)^{\frac{1}{\tilde{q}-\tilde{p}}},$$

and

$$I(\lambda u)=\lambda g'(\lambda)=\lambda\frac{dJ(\lambda u)}{d\lambda}\begin{cases}>0, & 0<\lambda<\lambda^*,\\ =0, & \lambda=\lambda^*,\\ <0, & \lambda^*<\lambda<+\infty.\end{cases}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let $u\in W_0^{1,p(x)}(\Omega)$, $\|\nabla u\|_{p(.)}\neq 0$, then*

(i) if $0<\|\nabla u\|_{p(.)}<r(\delta)=(\frac{\delta}{C_*^{q_2}})^{\frac{1}{q_2-p_1}}$, then $I_\delta(u)>0$;

(ii) $\|\nabla u\|_{p(.)}\geq r(\delta)$ provided that $I_\delta(u)\leq 0$.

Proof. (i) From Lemmas 2.1 and 2.3, we have

$$\begin{aligned}I_\delta(u)&=\delta\int_\Omega|\nabla u|^{p(x)}dx-\int_\Omega|u|^{q(x)}dx\geq\delta\|\nabla u\|_{p(.)}^{p_1}-\|u\|_{q(.)}^{q_2}\\ &\geq\delta\|\nabla u\|_{p(.)}^{p_1}-C_*^{q_2}\|\nabla u\|_{p(.)}^{q_2}=\|\nabla u\|_{p(.)}^{p_1}\Big(\delta-C_*^{q_2}\|\nabla u\|_{p(.)}^{q_2-p_1}\Big).\end{aligned}\tag{3.1}$$

Hence, when $0<\|\nabla u\|_{p(.)}<r(\delta)$, we have $I_\delta(u)>0$.

(ii) If $I_\delta(u)<0$, from (3.1) we can see that $\|\nabla u\|_{p(.)}^{p_1}(\delta-C_*^{q_2}\|\nabla u\|_{p(.)}^{q_2-p_1})<0$, which means

$$\|\nabla u\|_{p(.)}\geq\left(\frac{\delta}{C_*^{q_2}}\right)^{\frac{1}{q_2-p_1}}=r(\delta).$$

$I_\delta(u)=0$ gives

$$\delta\int_\Omega|\nabla u|^{p(x)}dx=\int_\Omega|u|^{q(x)}dx.$$

From Lemmas 2.1 and 2.3, we get

$$\delta\|\nabla u\|_{p(.)}^{p_1}\leq\delta\int_\Omega|\nabla u|^{p(x)}dx\leq\|u\|_{q(.)}^{q_2}\leq C_*^{q_2}\|\nabla u\|_{p(.)}^{q_2},$$

Consequently,

$$\|\nabla u\|_{p(.)}\geq\left(\frac{\delta}{C_*^{q_2}}\right)^{\frac{1}{q_2-p_1}}=r(\delta).\tag{3.2}$$

This completes the proof of the lemma. □

Lemma 3.3. *Let $I_\delta(u)=0$, then $J(u)\geq m(\delta)r(\delta)^{p_1}$ with $m(\delta)=\frac{1}{p_+}-\frac{\delta}{q_-}$, $0<\delta<\frac{q_-}{p_+}$. Moreover,*

$$d\geq\left(\frac{1}{p_+}-\frac{1}{q_-}\right)C_*^{\frac{p_1q_2}{p_1-q_2}}. \tag{3.3}$$

Proof. From Lemma 3.2, when $I_\delta(u)=0$, we have $\|\nabla u\|_{p(.)}\geq r(\delta)$. Therefore

$$\begin{aligned}J(u)&=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx\\&\geq\left(\frac{1}{p_+}-\frac{\delta}{q_-}\right)\int_\Omega|\nabla u|^{p(x)}dx+\frac{1}{q_-}I_\delta(u)\\&\geq\left(\frac{1}{p_+}-\frac{\delta}{q_-}\right)\|\nabla u\|_{p(.)}^{p_1}\geq\left(\frac{1}{p_+}-\frac{\delta}{q_-}\right)r(\delta)^{p_1}\\&=m(\delta)r(\delta)^{p_1}.\end{aligned} \tag{3.4}$$

Let $\delta=1$, by the definition of d, we have (3.3). □

Next, we define a function

$$\lambda=\lambda(\delta)=\left(\frac{\delta\int_\Omega|\nabla u|^{p(x)}dx}{\int_\Omega|u|^{q(x)}dx}\right)^{\frac{1}{\bar{q}-\bar{p}}}.$$

It is easy to check that $I_\delta(\lambda(\delta)u)=0$ for all $u\in W_0^{1,p(x)}$. For this λ, we get

$$\lim_{\delta\to0}\lambda(\delta)=0,\qquad\lim_{\delta\to+\infty}\lambda(\delta)=+\infty.$$

Furthermore,

$$\lim_{\delta\to0}J(\lambda u)=0,\quad\lim_{\delta\to+\infty}J(\lambda u)=-\infty,\ \text{ and }\ \lim_{\delta\to+\infty}d(\delta)=-\infty. \tag{3.5}$$

With the definition of $\lambda(\delta)$, we give the following results:

Lemma 3.4. *Let $d(\delta)$ be defined as before, $\|\nabla u\|_{p(.)}\neq0$, then*

(i) there exists a unique b such that $d(b)=0$, and $b\in[\frac{q_-}{p_+},\frac{q_+}{p_-}]$;

(ii) $d(\delta)$ is strictly increasing on $0<\delta<1$, strictly decreasing on $1<\delta<b$ and takes the maximum at $d=d(1)$.

Proof. (i) From Lemma 3.3, we know that $d(b)>0$ for $0<\delta<\frac{q_-}{p_+}$. On the other hand, from the definition of $d(\delta)$, one has,

$$\begin{aligned}d(\delta)\leq J(u)&\leq\frac{1}{p_-}\int_\Omega|\nabla u|^{p(x)}dx-\frac{1}{q_+}\int_\Omega|u|^{q(x)}dx\\&=\left(\frac{1}{p_-}-\frac{\delta}{q_+}\right)\int_\Omega|\nabla u|^{p(x)}dx.\end{aligned}$$

Then $d(\delta)<0$ when $\delta>\frac{q_+}{p_-}$. Hence there exists an b such that $d(b)=0$ with $b\in[\frac{q_-}{p_+},\frac{q_+}{p_-}]$.
(ii) We only need to prove that $d(\delta')<d(\delta'')$ for any $0<\delta'<\delta''<1$ or $1<\delta''<\delta'<b$, for any $u\in W_0^{1,p(x)}(\Omega)$, $I_{\delta''}(u)=0$, there exists $v\in W_0^{1,p(x)}(\Omega)$ and a constant $\varepsilon(\delta',\delta'')>0$, such that $I_{\delta'}(v)=0$ and $J(u)-J(v)>\varepsilon(\delta',\delta'')>0$.

In fact, for above u and the definition of $\lambda(\delta)$, $I_\delta(\lambda(\delta)u)=0$, $\lambda(\delta'')=1$.

$$\begin{aligned} g'(\lambda)=\frac{dJ(\lambda u)}{d\lambda}&=\int_\Omega \lambda^{p(x)-1}|\nabla u|^{p(x)}dx-\int_\Omega \lambda^{q(x)-1}|u|^{q(x)}dx\\ &=\frac{1}{\lambda}[\int_\Omega |\nabla(\lambda u)|^{p(x)}dx-\int_\Omega |\lambda u|^{q(x)}dx]\\ &=\frac{1}{\lambda}[(1-\delta)\int_\Omega |\nabla(\lambda u)|^{p(x)}dx+I_\delta(\lambda u)]\\ &=\frac{1-\delta}{\lambda}\int_\Omega |\nabla(\lambda u)|^{p(x)}dx. \end{aligned} \tag{3.6}$$

Taking $v=\lambda(\delta')u$, then $I_{\delta'}(v)=0$ and $\|\nabla v\|_{p(.)}\neq 0$. Since $\lambda(\delta)$ is increasing in δ, then for $0<\delta'<\delta''<1$ we have

$$\begin{aligned} J(u)-J(v)&=g(1)-g(\lambda(\delta'))=g(\lambda(\delta''))-g(\lambda(\delta'))\\ &=\int_{\lambda(\delta')}^{\lambda(\delta'')} g'(\lambda)d\lambda=\int_{\lambda(\delta')}^{\lambda(\delta'')}\frac{1-\delta}{\lambda}\int_\Omega |\nabla(\lambda u)|^{p(x)}dxd\lambda\\ &\geq \frac{1-\delta''}{\lambda(\delta'')}\int_{\lambda(\delta')}^{\lambda(\delta'')}\|\nabla(\lambda u)\|_{p(.)}^{p_1}d\lambda\\ &\geq \frac{1-\delta''}{\lambda(\delta'')}r(\delta')^{p_1}(\lambda(\delta'')-(\lambda(\delta'))=\varepsilon(\delta',\delta'')>0. \end{aligned} \tag{3.7}$$

Hence, $d(\delta')<d(\delta'')$. For $1<\delta''<\delta'<b$,

$$\begin{aligned} J(u)-J(v)&=\int_{\lambda(\delta')}^{\lambda(\delta'')}\frac{1-\delta}{\lambda}\int_\Omega |\nabla(\lambda u)|^{p(x)}dxd\lambda\\ &=\int_{\lambda(\delta'')}^{\lambda(\delta')}\frac{\delta-1}{\lambda}\int_\Omega |\nabla(\lambda u)|^{p(x)}dxd\lambda\\ &\geq \frac{\delta''-1}{\lambda(\delta')}r(\delta'')^{p_1}(\lambda(\delta')-(\lambda(\delta''))=\varepsilon(\delta',\delta'')>0. \end{aligned} \tag{3.8}$$

This completes the proof of the lemma. □

Lemma 3.5. *Let* $0<J(u)<d(1)=d$ *for some* $u\in W_0^{1,p(x)}(\Omega)$*, then there must exist two roots* δ_1,δ_2 *(*$\delta_1<1<\delta_2$*) of the equation* $d(\delta)=J(u)$*, and* $I_\delta(u)$ *is unchangeable for* $\delta\in(\delta_1,\delta_2)$*.*

Proof. From $J(u)>0$ and Lemma 3.4, there exists $\delta_1,\delta_2\in(0,b)$ such that $d(\delta_1)=d(\delta_2)<d$.

We assume $I_\delta(u)$ change sign on (δ_1,δ_2), then there exists $\overline{\delta}\in(\delta_1,\delta_2)$ such that $I_{\overline{\delta}}(u)=0$. With the definition of $d(\delta)$, $J(u)\geq d(\overline{\delta})$ which is contradictive with $J(u)=d(\delta_1)=d(\delta_2)<d(\overline{\delta})$. This completes the proof. □

Proposition 3.1. Let $u_0 \in W_0^{1,p(x)}(\Omega)$, $0<J(u_0)\le\sigma=d(\delta_1)=d(\delta_2)$, $I(u_0)>0$, if $u=u(x,t)$ is a weak solution of (1.1), then $u\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$.

Proof. For $\delta\in[1,\delta_2)$, we have

$$\begin{aligned} I_\delta(u_0) &= \delta\int_\Omega |\nabla u_0|^{p(x)}dx - \int_\Omega |u_0|^{q(x)}dx \\ &= (\delta-1)\int_\Omega |\nabla u_0|^{p(x)}dx + I(u_0)>0. \end{aligned} \tag{3.9}$$

From Lemma 3.5, we know that $I_\delta(u_0)$ is unchangeable on (δ_1,δ_2), then $I_\delta(u_0)>0$ for $\delta\in(\delta_1,\delta_2)$. This along with $0<J(u_0)\le\sigma<d(\delta)$ imply $u_0\in W_\delta$.

Multiplying the first equation of (1.1) by u_t and integrating the resulting equality over $\Omega\times[0,t]$, we have

$$\int_0^t\int_\Omega u_\tau^2 dxd\tau + J(u) = J(u_0) < d(\delta), \quad \text{for any } t\in[0,T),\ \delta\in(\delta_1,\delta_2), \tag{3.10}$$

where T is the maximal existence time of u. If $u\notin W_\delta$, then there exists $t_0\in(0,T)$ such that $u(x,t_0)\in\partial W_\delta$ for some $\delta\in(\delta_1,\delta_2)$, then

$$I_\delta(u(x,t_0))=0, \quad \|\nabla u(x,t_0)\|_{p(.)}\neq 0, \quad \text{or} \quad J(u(x,t_0))=d(\delta).$$

By (3.9), we know $J(u(x,t_0))=d(\delta)$ is false. Thus, we have $I_\delta(u(x,t_0))=0$ and $\|\nabla u(x,t_0)\|_{p(.)}$ $\neq 0$. However, by the definition of $d(\delta)$, one has $J(u(x,t_0))\ge d(\delta)$ which contradicts with (3.9). □

Proof of Theorem 3.1. We choose $\{W_j(x)\}$ as the orthogonal basis of $W_0^{1,p(.)}(\Omega)$ and construct the following approximate solution $u_m(x,t)$ of the problem (1.1) as was done in [2]. Let

$$u_m(x,t)=\sum_{j=1}^m g_j(t)W_j(x), \qquad m=1,2,\dots,$$

which satisfy

$$\int_\Omega u_{mt}W_s dx + \int_\Omega |\nabla u_m|^{p(x)-2}\nabla u_m \nabla W_s dx = \int_\Omega |u_m|^{q(x)-2}u_m W_s dx, \ \text{for } s=1,2\dots, \tag{3.11a}$$

$$u_m(x,0)=\sum_{j=1}^m a_j W_j(x)\to u_0 \ \text{ in } \ W_0^{1,p(.)}(\Omega),\ (m\to+\infty). \tag{3.11b}$$

Multiplying (3.11a) by $g_s'(t)$ and summing for s from 1 to m, for sufficiently large m, we obtain

$$\int_0^t\int_\Omega u_{m\tau}^2 dxd\tau + J(u_m(t)) = J(u_m(0)) < d, \qquad \text{for } 0<t<+\infty. \tag{3.12}$$

From the fact that $u_m(x,t)\in W$ (see the proof of Proposition 3.6), we gain

$$\begin{aligned} J(u_m) &= \int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}dx - \int_\Omega \frac{1}{q(x)}|u_m|^{q(x)}dx \\ &\geq \left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega |\nabla u_m|^{p(x)}dx + \frac{1}{q_-}I(u_m) \\ &> \left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega |\nabla u_m|^{p(x)}dx. \end{aligned} \tag{3.13}$$

Combing with (3.12), we get

$$\int_0^t \int_\Omega u_{m\tau}^2 dx d\tau + \left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega |\nabla u_m|^{p(x)}dx < d,$$

which implies

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau < d, \tag{3.14}$$

$$\|\nabla u_m\|_{p(.)} < \left(\frac{p_+ q_-}{q_- - p_+}d\right)^{\frac{1}{p_1}} = M. \tag{3.15}$$

Therefore

$$\|u_m\|_{q(.)} \leq C_* \|\nabla u_m\|_{p(.)} < C_* M. \tag{3.16}$$

Denote $\to^{W^*}$ as the weakly star convergence. Then from (3.14)-(3.16), there exist u and a subsequence still denoted as $\{u_m\}$ such that, as $m\to\infty$,

$$\begin{aligned} &u_m \to^{W^*} u && \text{in } L^\infty(0,\infty;W_0^{1,p(.)}(\Omega)), \\ &u_{mt} \to^{W^*} u_t && \text{in } L^\infty(0,\infty;L^2(\Omega)). \end{aligned}$$

Hence in (3.10), for s fixed and $m\to\infty$, we have

$$\int_\Omega u_t W_s dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla W_s dx = \int_\Omega |u|^{q(x)-2} u W_s dx. \tag{3.17}$$

On the other hand, from (3.11), we obtain $u(0,x)=u_0(x)$ in $W_0^{1,p(.)}(\Omega)$. Then u is a global weak solution of the problem (1.1). From Proposition 3.6, we know that the global weak solution $u\in W$. □

Proof of Theorem 3.2. Let $\mu_m = 1-\frac{1}{m}$ and $u_{0m}=\mu_m u_0$, $m=2,3...$, We consider the following problem:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q(x)-2}u, & (x,t)\in\Omega\times(0,T), \\ u=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_{0m}(x), & x\in\Omega. \end{cases} \tag{3.18}$$

From $I(u_0)\geq 0$ and Lemma 3.1, we have $\lambda=\lambda^*(u_0)\geq 1$. Therefore we can deduce that $I(u_{0m})>0$ and

$$J(u_{0m})\geq\left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega|\nabla u_{0m}|^{p(x)}dx+\frac{1}{q_-}I(u_{0m})>0. \tag{3.19}$$

On the other hand, we observe that

$$J(u_{0m})=J(\mu_m u_0)<J(u_0)=d.$$

From Theorem 3.1 that for each m, the problem (3.18) has a global weak solution

$$u_m\in L^\infty(0,\infty;W_0^{1,p(.)}(\Omega)),\;\; u_{mt}\in L^\infty(0,\infty;L^2(\Omega)) \text{ and } u_m\in W \;\text{ for } 0<t<\infty,$$

satisfying for any $v\in W_0^{1,p(.)}(\Omega)$, $0<t<\infty$,

$$\int_\Omega u_{mt}v dx+\int_\Omega|\nabla u_m|^{p(x)-2}\nabla u_m v dx=\int_\Omega|u_m|^{q(x)-2}u_m v dx, \tag{3.20}$$

$$\int_0^t\int_\Omega u_{m\tau}^2 dxd\tau+J(u_m(t))=J(u_{0m})<J(u_0)=d. \tag{3.21}$$

We also have

$$\begin{aligned} J(u_m)&\geq\left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega|\nabla u_m|^{p(x)}dx+\frac{1}{q_-}I(u_m)\\ &\geq\left(\frac{1}{p_+}-\frac{1}{q_-}\right)\int_\Omega|\nabla u_m|^{p(x)}dx. \end{aligned}\tag{3.22}$$

From (3.21) and (3.22) we can obtain (3.14)-(3.16). Then the following proof is similar to the later part of Theorem 3.1. □

4 Blow-up criterion

In this section, we discuss the blow-up solution of the problem (1.1). First, we show that the solution will blow up in finite time with $u_0\not\equiv 0$ and nonpositive initial energy functional $J(u_0)$. Then by defining an appropriate and positive function $F(t)$ on $[0,T]$ and using the method of concavity, we find an upper bound for the blow-up time. Assume $p(x)$ and $q(x)$ satisfy:

$$2<p_-\leq p(x)\leq p_+<q_-\leq q(x)\leq q_+<+\infty.$$

Define

$$F(t)=\int_0^t\int_\Omega u^2 dxd\tau+(T-t)\int_\Omega u_0^2 dx+a_1(t+a_2)^2, \tag{4.1}$$

where a_1,a_2 and T are some positive constants to be determined.

We now give the following results about blow-up.

Theorem 4.1. *Let u be a solution of* (1.1) *with* $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(.)}(\Omega)$, $u_0 \not\equiv 0$, $J(u_0) \le 0$. *Then the solution of the problem* (1.1) *blows up in finite time.*

Theorem 4.2. *Let u be a solution of* (1.1), *and F be given by* (4.1), *initial energy functional* $J(u_0)$ *and* a_1 *satisfy* $J(u_0) \le \frac{(1-q_-)a_1}{q_-}$. *Then for some positive constant γ satisfies* $0 < \gamma \le \frac{q_- - 2}{2}$,

(i) $(F^{-\gamma})''(t) \le 0$ *for all* $t \in [0,T]$;

(ii) the solution will blow up at finite time, and is bounded from up by

$$\frac{a_1 a_2^2}{2\gamma a_1 a_2 - \int_\Omega u_0^2 dx} \quad \textit{with} \;\; a_2 > \frac{1}{2\gamma a_1} \int_\Omega u_0^2 dx.$$

Proof of Theorem 4.1. Let $u(x,t)$ be a solution of (1.1). Suppose $G(t) = \frac{1}{2}\int_0^t \int_\Omega u^2 dx d\tau$. Then

$$G'(t) = \frac{1}{2}\int_\Omega u^2 dx; \tag{4.2}$$

$$\begin{aligned}
G''(t) &= \int_\Omega u u_t dx = \int_\Omega u[\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q(x)-2}u]dx \\
&= -\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{q(x)} dx \\
&> -p_+\left[\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx\right] = -p_+ J(u).
\end{aligned} \tag{4.3}$$

By a simple calculation we get

$$\frac{dJ(u)}{dt} = -\int_\Omega u_t^2 dx. \tag{4.4}$$

Integrating (4.4) from 0 to t, we have

$$J(u(t)) = J(u_0) - \int_0^t \int_\Omega u_\tau^2 dx d\tau. \tag{4.5}$$

Substitute (4.5) into (4.3), we obtain

$$G''(t) > p_+ \int_0^t \int_\Omega u_\tau^2 dx d\tau - p_+ J(u_0) \ge p_+ \int_0^t \int_\Omega u_\tau^2 dx d\tau. \tag{4.6}$$

On the other hand,

$$\begin{aligned}
G'(t) - G'(0) &= \frac{1}{2}\int_\Omega u(x,t)^2 dx - \frac{1}{2}\int_\Omega u_0^2 dx = \int_0^t \int_\Omega u u_\tau dx d\tau \\
&\le \left(\int_0^t \int_\Omega u^2 dx d\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega u_\tau^2 dx d\tau\right)^{\frac{1}{2}} \le \left(\frac{2}{p_+}\right)^{\frac{1}{2}} G(t)^{\frac{1}{2}} G''(t)^{\frac{1}{2}}.
\end{aligned} \tag{4.7}$$

Consequently, we have

$$\frac{p_+}{2}(G'(t)-G'(0))^2 \le G(t)G''(t). \tag{4.8}$$

Next, we assume for contradiction that the solution u exists for all $t>0$, and claim that

$$\int_0^{t_0}\int_\Omega (u_\tau)^2 dxd\tau > 0, \quad \text{for any } t_0>0. \tag{4.9}$$

Otherwise, there exists a $t_0>0$ such that

$$\int_0^{t_0}\int_\Omega (u_\tau)^2 dxd\tau = 0, \tag{4.10}$$

and hence $u_t=0$ for all $t\in(0,t_0]$. Then from (4.5) we have

$$J(u)=J(u_0), \quad \text{for } x\in\Omega,\ t\in(0,t_0]. \tag{4.11}$$

From (4.3) we get

$$-p_+J(u)<0, \quad \text{for } x\in\Omega,\ t\in(0,t_0]. \tag{4.12}$$

This implies $J(u)>0$ for $t\in(0,t_0]$, which contradicts with (4.11). Integrating (4.6) from t_0 to t , we have

$$G'(t)>G'(t_0)+p_+\int_{t_0}^t\int_0^T\int_\Omega u_\tau^2 dxdsd\tau, \tag{4.13}$$

which implies

$$\lim_{t\to\infty} G'(t)=+\infty. \tag{4.14}$$

Then there exist $t^*\ge t_0$ and a positive constant α ($\alpha\in(1,\frac{p_+}{2})$), such that

$$\alpha(G'(t))^2 \le \frac{p_+}{2}(G'(t)-G'(0))^2 \le G(t)G''(t). \tag{4.15}$$

We denote $H(t)=(G(t))^{-\beta}$,$(0<\beta\le\alpha-1)$. Then $H(t)\to 0$ as $t\to\infty$, with $G(t^*)>0$. However, a simple computation with the above inequality yields

$$\begin{aligned} H''(t) &= \beta(G(t))^{-\beta-2}[(\beta+1)(G'(t))^2-G(t)G''(t)] \\ &\le \beta(G(t))^{-\beta-2}[\alpha(G'(t))^2-G(t)G''(t)] \le 0, \end{aligned} \tag{4.16}$$

for all $t\ge t^*$, a contradiction. □

Proof of Theorem 4.2. (i) By direct differentiation, we have

$$\begin{aligned}F'(t)&=\int_\Omega u^2dx-\int_\Omega u_0^2dx+2a_1(t+a_2)\\&=2\int_0^t\int_\Omega uu_\tau dxd\tau+2a_1(t+a_2)\\&=-2\int_0^t I(u)d\tau+2a_1(t+a_2),\end{aligned}\tag{4.17}$$

Consequently,

$$F''(t)=-2I(u)+2a_1.\tag{4.18}$$

From (4.5), we have

$$\begin{aligned}I(u)&=\int_\Omega|\nabla u|^{p(x)}dx-\int_\Omega|u|^{q(x)}dx\\&\le q_-\left[\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx\right]\\&=q_-J(u)=q_-J(u_0)-q_-\int_0^t\int_\Omega u_\tau^2dxd\tau.\end{aligned}\tag{4.19}$$

Then the equation for $F''(t)$ is reduced to

$$\begin{aligned}F''(t)&\ge-2q_-J(u_0)+2q_-\int_0^t\int_\Omega u_\tau^2dxd\tau+2a_1\\&\ge4(1+\gamma)\left[\int_0^t\int_\Omega u_\tau^2dxd\tau+a_1\right]-2q_-J(u_0)-2(1+2\gamma)a_1\\&\ge4(1+\gamma)\left[\int_0^t\int_\Omega u_\tau^2dxd\tau+a_1\right].\end{aligned}\tag{4.20}$$

It follows from (4.1), (4.17) and (4.20) that

$$\begin{aligned}&FF''-(1+\gamma)(F')^2\\\ge&4(1+\gamma)\left(\int_0^t\int_\Omega u_\tau^2dxd\tau+a_1\right)\left(\int_0^t\int_\Omega u^2dxd\tau+a_1(t+a_2)^2\right)\\&-4(1+\gamma)\left(\int_0^t\int_\Omega uu_\tau dxd\tau+a_1(t+a_2)\right)^2.\end{aligned}\tag{4.21}$$

It is easily seen by the Schwartz inequality that

$$FF''-(1+\gamma)(F')^2\ge0.\tag{4.22}$$

Since by direct computation

$$(F^{-\gamma})''(t)=-\gamma F^{-\gamma-2}[FF''-(1+\gamma)(F')^2].\tag{4.23}$$

It follows from (4.22) and (4.23) that $(F^{-\gamma})''(t) \leq 0$ for $t \in [0,T]$.

(ii) The result in (i) leads to

$$(F^{-\gamma})'(t) \leq (F^{-\gamma})'(0) = -\gamma F^{-\gamma-1}(0)F'(0). \tag{4.24}$$

An integration of the above inequality gives

$$F^{-\gamma}(t) \leq F^{-\gamma}(0) - \gamma F^{-\gamma-1}(0)F'(0)t = F^{-\gamma-1}(0)(F(0) - \gamma F'(0)t), \tag{4.25}$$

which means for some

$$T \leq \frac{F(0)}{\gamma F'(0)} = \frac{a_1 a_2^2}{2\gamma a_1 a_2 - \int_\Omega u_0^2 dx}, \quad F^{-\gamma}(t) \to 0 \;\; as \; t \to T.$$

This completes the proof. □

Remark 4.1. We know that the solution of the problem (1.1) will blow up in finite under some appropriate assumptions, that is $\lim_{t\to T^-} \|u(\cdot,t)\|_2 = +\infty$. From Lemma 2.4, we also obtain $\lim_{t\to T^-} \|u(\cdot,t)\|_{p(.),\Omega} = +\infty$.

Remark 4.2. If the conditions on $p(x)$ and $q(x)$ are replaced by $1 < p_- \leq p(x) \leq p_+$, and $max\{p_+,2\} < q_- \leq q(x) \leq q_+ < +\infty$, the results in Theorem 4.2 still be established.

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