J. Math. Study
doi: 10.4208/jms.v52n2.19.02

Vol. **52**, No. 2, pp. 127-159
June 2019

Solvability of the Nonlocal Initial Value Problem and Application to Design of Controller for Heat-equation with Delay

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Received September 31, 2018; Accepted April 11, 2019

Abstract. In this paper, we study the solvability of a distribution-valued heat equation with nonlocal initial condition. Under proper assumption on parameters we get the explicit solution of the distribution-valued heat equation. As an application, we further consider the stabilization problem of heat equation with partial-delay in internal control. By the parameterization design of feedback controller, we show if the integral kernel functions are determined by the solution of the distribution heat equation with nonlocal initial value problem, then the closed-loop system can be transformed into a system which is called the target system of the exponential stability under the bounded linear transformation. By selecting different distribution-valued kernel functions, we give the inverse transformation. Hence the closed-loop system is equivalent to the target system.

AMS subject classifications: 93D15, 93C20, 35B35

Key words: Abstract heat equation, Solvability, nonlocal initial value condition, internal delayed control, integral-type feedback controller, exponential stability.

1 Introduction

In this paper we consider the abstract heat-equation in space $\mathbb{X}$ of distributions (see, [1, Appendix E], also see Definition 2.1 in next section)

$$\begin{cases} \gamma_s(s,y)=\gamma_{yy}(s,y)+\rho\gamma(s,y), & y\in(0,1),\ s>0,\\ \gamma_y(s,1)=\gamma_y(s,0)=0, & s>0,\\ \beta\gamma(0,y)+\alpha\gamma(\tau,y)=\gamma_0(y), & y\in[0,1], \end{cases} \tag{1.1}$$

where ρ, α and β are real numbers, and $\gamma(s,y)$ is an abstract function defined on $\mathbb{R}_+\times[0,1]$ and valued in $\mathbb{X}$.

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If $\mathbb{X}=\mathbb{R}$ is real number set, and $\alpha=0$, $\beta=1$, then (1.1) is just the classical heat-equation with Neumann boundary conditions, which is the initial and boundary value problem. In this aspect, we refer to [2–4]. When $\alpha\neq 0$ and $\beta\neq 0$, it is a nonlocal initial and boundary value problem. For example, see [5–8] on the nonlocal initial value problem of Partial-Differential Equations.

If $\mathbb{X}$ is a Banach space, and $\alpha=0$, $\beta=1$, (1.1) is an abstract heat-equation with Neumann boundary conditions, which can be solved by the general approach such as C_0-semigroup theory of bounded linear operators. When $\beta\neq 0$ and $\alpha\neq 0$, the nonlocal initial value problem can be solved by C_0-semigroup theory for linear case, and by the fixed point theory for nonlinear case. In this case we refer to [9, 10].

If $\mathbb{Y}$ is a space of test function, and $\mathbb{X}=\mathbb{Y}'$ is the dual space, which is called the distribution space, (1.1) is an abstract heat-equation of the Neumann boundary conditions with valued-distribution. In this paper, we shall take $\mathbb{Y}=C_0^\infty(0,1)$, and $\mathbb{X}=\mathbb{Y}'$. More precisely, we need to solve the following partial differential equations

$$\begin{cases} \gamma_s(x,s,y)=\gamma_{yy}(x,s,y)+\rho\gamma(x,s,y), & x,y\in[0,1],\ s>0,\\ \gamma_y(x,s,1)=\gamma_y(x,s,0)=0, & x\in[0,1],\ s>0,\\ \beta\gamma(x,0,y)+\alpha\gamma(x,\tau,y)=-(\lambda+\rho)\delta(y-x), & x,y\in[0,1], \end{cases} \tag{1.2}$$

where λ, ρ are positive constants, and α and β are real numbers. Such a problem comes from the control theory (see, Section 3), in which our purpose is to design a state feedback control $u(x,t)$ to stabilize the heat system with delayed control

$$\begin{cases} w_t(x,t)=w_{xx}(x,t)+\rho w(x,t)+\alpha u(x,t)+\beta u(x,t-\tau), & x\in(0,1),\ t>0,\\ w_x(1,t)=w_x(0,t)=0, & t>0,\\ w(x,0)=w_0(x), & x\in[0,1],\\ u(x,t-\tau)=h(x,t), & t\in[0,\tau]. \end{cases} \tag{1.3}$$

The control and stabilization problem of heat equation or the system coupled with heat equation has been a hot topic in the control field. For instance, Zhao and Wang in [11] studied the stabilization of a coupled heat-ODE system; Li *et al.* in [12] investigated the rapid stabilization of heat equation in non-cylindrical domain; Baudouin *et al.* in [13] discussed the stability analysis of a system coupled to a heat equation. However, these papers do not consider the delay problem of control input. For ODE system, Artstein [14] gave a nice approach to deal with the delayed control. For the distributed delay, Bekiaris *et al.* [15, 16] gave a feedback control design based on the Lyapunov function approach. For the infinite dimensional system, the feedback controller might be sensitive for a small time delay, for example, see [17, 18]. In particular, when the control is of the form as (1.3), the feedback control design becomes a challenge work. Here we refer to the books [19] by Krstic and [20] by Ammari and Nicaise. For delay control problem, there are some important works for hyperbolic systems, for example, [21] for the wave equation, and [23] for wave network under the collocated feedback control; [22] for an Euler-Bernoulli beam, [24] for 1-d wave equation and [25] for Timoshenko beam under