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Müntz Spectral Method for Two-Dimensional Space-Fractional Convection-Diffusion Equation

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Abstract. In this paper, we propose and analyze a Müntz spectral method for a class of two-dimensional space-fractional convection-diffusion equations. The proposed methods make new use of the fractional polynomials, also known as Müntz polynomials, which can be regarded as continuation of our previous work. The extension is twofold. Firstly, the existing Müntz spectral method for fractional differential equation with fractional derivative order $0<\mu<1$ is generalized to $0<\mu\leq 2$, which is nontrivial since the classical Müntz polynomials only have no more than $H^{1/2+\lambda}$ regularity, where $0<\lambda\leq 1$ is the characteristic parameter of the Müntz polynomial. Secondly, 1D Müntz spectral method is extended to the 2D space-fractional convection-diffusion equation. Compared to the time-fractional diffusion equation, some new operators such as suitable H^1-projectors are needed to analyze the error of the numerical solution. The main contribution of the present paper consists of an efficient method combining the Crank-Nicolson scheme for the temporal discretization and a new spectral method using the Müntz Jacobi polynomials for the spatial discretization of the 2D space-fractional convection-diffusion equation. A detailed convergence analysis is carried out, and several error estimates are established. Finally a series of numerical experiments is performed to verify the theoretical claims.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

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1 Introduction

In the past two decades, fractional differential equations (FDEs) have gained increasing popularity, mainly due to its concrete applications in variety of fields such as control the-

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ory, biology, electro-chemical processes, viscoelastic materials, polymers, finance, etc. On the other side, interesting features of FDEs have also been attracting much attention from theoretical and numerical researchers. There exists an enormous amount of literature on numerical methods for FDEs and it is impossible to give a complete list of references. Among the existing methods, we mention [1,10–12,22,25,26,35,47] for various time discretizations for the time-fractional diffusion equation. Concerning space-fractional differential equations, we refer to [6,8,24,29,30,37–40,42,49] and the references therein for a variety of developed numerical methods, covering from finite difference methods to finite element methods.

It has become well-known now that solving fractional differential equations is accompanying a number of features/difficulties, including the non-locality of fractional differential operators, singularity/low regularity of the kernel functions and the solutions of the associated problems. The first feature unavoidably results in high storage cost for traditional approaches. This difficulty has led to the investigation of fast evaluation techniques using the approximation to the weakly singular kernel function by a sum-of-exponentials, resulting in some stepping schemes with reduced storage; see, e.g., Baffet and Hesthaven [2,3], Jiang et al. [17,48], Yan et al. [41], Zeng et al. [46], etc.

High order methods have also been considered as an efficient way to reduce the memory requirement. This consideration has motivated several researches, e.g., [13,15,16,18–20,27,28,33,34,45], in which various spectral methods have been constructed for time-fractional, space-fractional, or time-space-fractional differential equations. These methods could be highly efficient if the exact solution is smooth since the exponential convergence can be expected in this case. However, as it is aforementioned, the solution of FDEs is usually non-smooth due to the presence of the weakly singular kernel function in the definitions of fractional operators. Several other work have targeted at spectral methods for non smooth solutions. Among these methods, Zheng et al. [50] proposed a spectral method for the multi-term time-fractional diffusion equation. Zayernouri et al. [43] considered a Petrov-Galerkin spectral method using the so-called polyfractonomials, introduced in Zayernouri and Karniadakis [44]. Chen et al. [5] used generalized Jacobi functions to construct Petrov-Galerkin methods for a class of fractional initial/boundary value problems. In particular, Hou et al. [13, 15] introduced a general framework using Müntz polynomials for some weakly singular integro-differential equations and fractional differential equations. Both numerical experiments and theoretical analysis have shown that the computed solution by the Müntz spectral method can be exponentially accurate for a large class of fractional differential equations, even if the exact solution of these equations is not smooth.

The present work can be regarded as continuation of [13,15]. This extension is motivated by the fact that the singularity caused by the 2D domain corners has form similar to $(x^2+y^2)^\rho$, x^ρ, y^ρ, or their combination for some real number ρ, which could be efficiently approximated by Müntz polynomials. Note that the order of the fractional derivatives in our previous work [13, 15] is not more than 1, thus the Müntz spectral method can be easily applied for those problems for the reason that the classical Müntz polynomials with characteristic parameter λ belong to $H^{1/2+\lambda}$. However, space-fractional convection-diffusion equation is a class of problem with fractional derivative order from 0 to 2, hence the classical Müntz spectral method can't be directly used for it. To overcome this difficulty, we develop a new kind of Müntz Jacobi polynomials and study its fundamental approximation property. More precisely, we have obtained the optimal error estimates for its H^1-projector, which will play a key role in the error analysis for the new Münz spectral method to be constructed for the two-dimensional space-fractional convection-diffusion equation.

The main contribution of the present paper consists of an efficient method combining the Crank-Nicolson scheme for the temporal discretization and a new spectral method using the Müntz Jacobi polynomials for the spatial discretization for solving the 2D FCDE. An error analysis is carried out for the proposed method by overcoming the aforementioned difficulties. The paper is organized as follows: In the next section, we give some notations and present minimum preparatory materials of Müntz Jacobi polynomials. In Section 3, we present a second order temporal discretization for the 2D FCDE, and carry out an error analysis for the semi-discrete scheme. Section 4 is devoted to construct the Müntz spectral method for the spatial approximation of the semi-discrete problem. The optimal error estimates are first derived for the H_0^1-projector in both H^1-semi-norm and L^2-norm, then for the numerical solution. The obtained error estimate shows the spectral accuracy of the proposed Müntz spectral method for some kind of non-smooth solutions. The implementation details are provided in Section 5. We give some numerical examples in Section 6 to validate the proposed method. Finally we give some concluding remarks in the final section.

2 Problem and preparation

Let $\Lambda=(0,1)$ and $I=(0,T)$ be the space and time domain, respectively. We denote $\Omega:=\Lambda\times\Lambda$. In this paper we focus on the two-dimensional space-fractional convection-diffusion equation (FCDE):

$$\begin{cases} \partial_t u-\nu\Delta u+p\,{}_0D_x^{2\mu_1}u+q\,{}_0D_y^{2\mu_2}u=f(x,y,t), & (x,y,t)\in\Omega\times I,\\ u(x,y,0)=u_0(x,y), & (x,y)\in\Omega,\\ u|_{\partial\Omega}=0, & t\in I, \end{cases} \tag{2.1}$$

where $\nu\ge 0$, ${}_0D_x^{2\mu_1}$ and ${}_0D_y^{2\mu_2}$ denote the fractional derivative of order $2\mu_1$ and $2\mu_2$ respectively, with $0\le\mu_1,\mu_2<1,\mu_1,\mu_2\neq\frac{1}{2}$ for $\nu>0$ and $\frac{1}{2}<\mu_1,\mu_2<1$ for $\nu=0$. The coefficients p,q satisfy

$$p:=\begin{cases} \ge 0, & 0\le\mu_1<\dfrac{1}{2},\\ \le 0, & \dfrac{1}{2}<\mu_1<1, \end{cases} \qquad q:=\begin{cases} \ge 0, & 0\le\mu_2<\dfrac{1}{2},\\ \le 0, & \dfrac{1}{2}<\mu_2<1. \end{cases} \tag{2.2}$$

Here $f\in L^\infty(I;H^{-1}(\Omega))$ if $\nu>0$, and $f\in L^\infty(I;H^{-\mu_1,-\mu_2}(\Omega))$ if $\nu=0$, where $H^{-\mu_1,-\mu_2}(\Omega)$ denotes the dual space of $H_0^{\mu_1,\mu_2}(\Omega)$, to be specified later.

2.1 Fractional derivatives and Sobolev spaces

Firstly, we give some basic notations [7,31]. In what follows, c,c_0,c_1, etc are generic positive constants independent of any functions and of any discretization parameters. We use the expression $A\lesssim B$ to mean that $A\leq cB$.

We use ${}^{RL}_{\ a}D^\mu_x$ and ${}^{RL}_{\ x}D^\mu_b$, $x\in(a,b)$, to denote respectively the left and right Riemann-Liouville fractional derivatives of order $\mu\in(0,\infty)$, defined by

$$ {}^{RL}_{\ a}D^\mu_x v(x)=\frac{1}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_a^x (x-s)^{n-\mu-1}v(s)ds, \tag{2.3}$$

$$ {}^{RL}_{\ x}D^\mu_b v(x)=\frac{(-1)^n}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_x^b (s-x)^{n-\mu-1}v(s)ds, \tag{2.4}$$

where $\Gamma(\cdot)$ is the Gamma function and n is the integer number such that $n-1\leq\mu<n$.

Similarly, ${}^{C}_{a}D^\mu_x$ and ${}^{C}_{x}D^\mu_b$ are used to denote the left and right Caputo fractional derivatives respectively, defined as

$$ {}^{C}_{a}D^\mu_x v(x)=\frac{1}{\Gamma(n-u)}\int_a^x (x-s)^{n-\mu-1}v^{(n)}(s)ds, \tag{2.5}$$

$$ {}^{C}_{x}D^\mu_b v(x)=\frac{(-1)^n}{\Gamma(n-\mu)}\int_x^b (s-x)^{n-\mu-1}v^{(n)}(s)ds, \tag{2.6}$$

for $\mu\in(0,\infty)$, $n-1\leq\mu<n$, where $v^{(n)}$ is the n-th order derivative of v.

The Riemann-Liouville fractional derivatives and the Caputo ones are linked by the following relationship

$$ {}^{C}_{a}D^\mu_x v(x)={}^{RL}_{\ a}D^\mu_x v(x)-\sum_{k=0}^{n-1}\frac{v^{(k)}(a)}{\Gamma(k+1-\mu)}x^{k-\mu}. \tag{2.7}$$

The integral operators ${}_aI^\mu_x$ and ${}_xI^\mu_b,\mu\in(0,\infty)$, are defined by

$$ \begin{aligned} {}_aI^\mu_x v(x)&=\frac{1}{\Gamma(\mu)}\int_a^x (x-s)^{\mu-1}v(s)ds,\\ {}_xI^\mu_b v(x)&=\frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}v(s)ds. \end{aligned} \tag{2.8}$$

Lemma 2.1. ([5, Lemma 2.4]) *Let $\mu>0,x\in(-1,1)$, and m be nonnegative integer number. Then (i) for $\alpha\in R,\beta>-1$, it holds*

$$ {}_{-1}I^\mu_x\{(1+x)^\beta J_m^{\alpha,\beta}(x)\}=\frac{\Gamma(m+\beta+1)}{\Gamma(m+\beta+\mu+1)}(1+x)^{\beta+\mu}J_m^{\alpha-\mu,\beta+\mu}(x); \tag{2.9}$$

(ii) for $\alpha>-1,\beta\in R$*, it holds*

$$ {}_xI_1^\mu\{(1-x)^\alpha J_m^{\alpha,\beta}(x)\}=\frac{\Gamma(m+\alpha+1)}{\Gamma(m+\alpha+\mu+1)}(1-x)^{\alpha+\mu}J_m^{\alpha+\mu,\beta-\mu}(x). \tag{2.10}$$

Although using the Caputo definition for ${}_0D_x^{2\mu_1}$ and ${}_0D_y^{2\mu_2}$ in problem (2.1) allows handing inhomogeneous boundary conditions in an easier way, we choose to consider the Riemann-Liouville definition throughout this paper for the reason that most of the useful results have been established for the Riemann-Liouville definition. In the following, we introduce several fractional spaces to be used in the paper.

For any real $s>0$, we define the left-fractional derivative space [9,19,20]:

$$ {}^lH^s(\Lambda):=\{v|\,v\in L^2(\Lambda),{}_0D_x^sv\in L^2(\Lambda)\},$$

equipped with the semi-norm

$$|v|_{{}^lH^s(\Lambda)}:=\|{}_0D_x^sv\|_{0,\Lambda},$$

and the norm

$$\|v\|_{{}^lH^s(\Lambda)}:=\left(\|v\|_{0,\Lambda}^2+|v|_{{}^lH^s(\Lambda)}^2\right)^{1/2}.$$

Let the space ${}^lH_0^s(\Lambda)$ be

$$ {}^lH_0^s(\Lambda):=\overline{C_0^\infty(\Lambda)}^{\|\cdot\|_{{}^lH^s(\Lambda)}},$$

where

$$C_0^\infty(\Lambda)=\{v|\,v\in C^\infty(\Lambda)\text{ having compact support in }\Lambda\}.$$

Similarly we define the right-fractional derivative space:

$$ {}^rH^s(\Lambda):=\{v|\,v\in L^2(\Lambda),{}_xD_1^sv\in L^2(\Lambda)\},$$

equipped with the semi-norm

$$|v|_{{}^rH^s(\Lambda)}:=\|{}_xD_1^sv\|_{0,\Lambda},$$

and the norm

$$\|v\|_{{}^rH^s(\Lambda)}:=\left(\|v\|_{0,\Lambda}^2+|v|_{{}^rH^s(\Lambda)}^2\right)^{1/2}.$$

Let ${}^rH_0^s(\Lambda)$ be the space

$$ {}^rH_0^s(\Lambda):=\overline{C_0^\infty(\Lambda)}^{\|\cdot\|_{{}^rH^s(\Lambda)}}.$$

Furthermore, we recall some classical Sobolev spaces: for real $s\geq 0$,

$$H^s(\mathbb{R}):=\{v|\,v\in L^2(\mathbb{R});\,(1+|\omega|^2)^{s/2}\mathcal{F}(v)(\omega)\in L^2(\mathbb{R})\},$$

endowed with the semi-norm

$$|v|_{s,\mathbb{R}}=\big\||\omega|^s\mathcal{F}(v)(\omega)\big\|_{0,\mathbb{R}},$$

and the norm

$$\|v\|_{s,\mathbb{R}}=\big\|(1+|\omega|^2)^{s/2}\mathcal{F}(v)(\omega)\big\|_{0,\mathbb{R}},$$

where $\mathcal{F}(v)$ denotes the Fourier transform of v.

The space $H^s(\Lambda)$ for bounded domain Λ is defined by:

$$H^s(\Lambda):=\big\{v\in L^2(\Lambda)\,|\,\exists\tilde{v}\in H^s(\mathbb{R}) \text{ such that } \tilde{v}|_\Lambda=v\big\},$$

equipped with the semi-norm and norm as follows:

$$|v|_{s,\Lambda}=\inf_{\tilde{v}\in H^s(\mathbb{R}),\tilde{v}|_\Lambda=v}|\tilde{v}|_{s,\mathbb{R}},\quad \|v\|_{s,\Lambda}=\inf_{\tilde{v}\in H^s(\mathbb{R}),\tilde{v}|_\Lambda=v}\|\tilde{v}\|_{s,\mathbb{R}}.$$

Let $H_0^s(\Lambda)$ denote the closure of $C_0^\infty(\Lambda)$ with respect to the norm $\|\cdot\|_{s,\Lambda}$.

For any function v in $H^s(\Lambda)$, we define

$$|v|^*_{s,\Lambda}:=|({}_aD_t^s v,{}_tD_b^s v)_\Lambda|^{1/2},\quad \|v\|^*_{s,\Lambda}:=\big(\|v\|^2_{0,\Lambda}+|v|^{*2}_{s,\Lambda}\big)^{1/2}. \tag{2.11}$$

It has been proved, see Proposition 2.3 in [13], that $|\cdot|^*_{s,\Lambda}$ and $\|\cdot\|^*_{s,\Lambda}$ is respectively a semi-norm and norm in $H_0^s(\Lambda)$ for all real $s>0$, $s\neq n+1/2$, $n\in\mathbb{N}$. Moreover, we have

Lemma 2.2 (see [13,19,20]). *Let $s>0$, $s\neq n+\frac{1}{2}$, $n\in\mathbb{N}$. The spaces ${}^lH_0^s(\Lambda)$, ${}^rH_0^s(\Lambda)$, and $H_0^s(\Lambda)$ are equal and their semi-norms and norms are equivalent to $|\cdot|^*_{s,I}$ and $\|\cdot\|^*_{s,I}$ respectively.*

The following formulas are useful; see, e.g., [19, 20]: for real s, $0<s\leq 1$, $s\neq\frac{1}{2}$, and $w,v\in H_0^s(\Lambda)$, it holds

$$({}_0D_x^{2s}w,v)=({}_0D_x^s w,{}_xD_1^s v).$$

For all $s>0, v\in C_0^\infty(\mathbb{R})$, we have

$$({}_{-\infty}D_x^s v,{}_xD_{+\infty}^s v)_\mathbb{R}=\cos(\pi s)\|{}_{-\infty}D_x^s v\|^2_{0,\mathbb{R}},\quad ({}_{-\infty}D_x^s v,{}_xD_{+\infty}^s v)_\mathbb{R}=\cos(\pi s)\|{}_xD_{+\infty}^s v\|^2_{0,\mathbb{R}}.$$

For bounded (a,b) and any $v\in H_0^s((a,b))$, let $\tilde{v}$ be the extension of v by zero outside of (a,b). Then we have

$$\text{supp}(\tilde{v})\subset(a,b),\quad \text{supp}({}_{-\infty}D_x^s\tilde{v})\subset(a,\infty),\quad \text{supp}({}_xD_\infty^s\tilde{v})\subset(-\infty,b).$$

Therefore the product ${}_{-\infty}D_x^s\tilde{v}\,{}_xD_\infty^s\tilde{v}$ has support in (a,b). It follows that

$$({}_aD_x^s v,{}_xD_b^s v)_{(a,b)}=({}_{-\infty}D_x^s\tilde{v},{}_xD_{+\infty}^s\tilde{v})_\mathbb{R}=\cos(\pi s)\|{}_aD_x^s\tilde{v}\|^2_{0,(a,+\infty)}=\cos(\pi s)\|{}_xD_b^s\tilde{v}\|^2_{0,(-\infty,b)}.$$

Lemma 2.3 (Fractional Poincaré-Friedrichs Inequality; see [9]). *For $\mu>0$, $v\in H_0^\mu((a,b))$, we have*

$$\|v\|_{0,(a,b)}\lesssim|v|_{H_0^\mu((a,b))}.$$

Moreover, if $0<s<\mu$, $s\neq n-1/2$, $n\in\mathbb{N}$, it holds

$$|v|_{H_0^s((a,b))}\lesssim|v|_{H_0^\mu((a,b))}.$$

In the error estimation, we will make use of a number of Sobolev spaces, including the non-uniform Jacobi-weighted Sobolev spaces, defined by:

$$B^m_{\omega^{\alpha,\beta,1}}(\Lambda):=\{v:\partial_x^k v\in L^2_{\omega^{\alpha+k,\beta+k,1}}(\Lambda),\ 0\le k\le m\},\quad m\in\mathbb{N}.$$

equipped with the inner product, norm, and semi-norm respectively as follows:

$$(u,v)_{B^m_{\omega^{\alpha,\beta,1}}}=\sum_{k=0}^{m}(\partial_x^k u,\partial_x^k v)_{\omega^{\alpha+k,\beta+k,1}},\quad \|v\|_{m,\omega^{\alpha,\beta,1}}=(v,v)^{1/2}_{B^m_{\omega^{\alpha,\beta,1}}},\quad |v|_{m,\omega^{\alpha,\beta,1}}=\|\partial_x^m v\|_{0,\omega^{\alpha+m,\beta+m,1}},$$

where $\omega^{\alpha,\beta,1}(x)=(1-x)^\alpha x^\beta$ with $\alpha,\beta\ge -1$.

For a Sobolev space X, equipped with the norm $\|\cdot\|_X$, we denote

$$\begin{aligned}
&H^s(\Lambda;X):=\{v|\ \|v(x,\cdot)\|_X\in H^s(\Lambda)\},\quad s\ge 0,\\
&H^s_0(\Lambda;X):=\{v|\ \|v(x,\cdot)\|_X\in H^s_0(\Lambda)\},\quad s\ge 0,\\
&B^m_{\omega^{\alpha,\beta,1}}(\Lambda;X):=\{v|\ \|v(x,\cdot)\|_X\in B^m_{\omega^{\alpha,\beta,1}}(\Lambda)\},\quad m\in\mathbb{N},
\end{aligned}$$

endowed respectively with the norm:

$$\|v\|_{H^s(\Lambda;X)}:=\|\|v(x,\cdot)\|_X\|_{s,\Lambda},\quad \|v\|_{B^m_{\omega^{\alpha,\beta,1}}(\Lambda;X)}:=\|\|v(x,\cdot)\|_X\|_{B^m_{\omega^{\alpha,\beta,1}}}.$$

When X is $H^\sigma(\Lambda)$ or $H^\sigma_0(\Lambda)$, $\sigma\ge 0$, the norm of the space $H^s(\Lambda;X)$ will be denoted by $\|\cdot\|_{s,\sigma,\Omega}$.

We also define the space:

$$H^{s,\sigma}_0(\Omega):=H^s_0(\Lambda;L^2(\Lambda))\cap L^2(\Lambda;H^\sigma_0(\Lambda)),$$

equipped with the norm

$$\|v\|_{H^{s,\sigma}_0}:=(\|v\|^2_{H^s_0(\Lambda;L^2(\Lambda))}+\|v\|^2_{L^2(\Lambda;H^\sigma_0(\Lambda))})^{1/2}=(\|v\|^2_{s,0}+\|v\|^2_{0,\sigma})^{1/2}. \tag{2.12}$$

It can be verified that $H^{s,\sigma}_0(\Omega)$ is a Banach space. In particular, if $s=\sigma$, then $H^{s,\sigma}_0(\Omega)=H^s_0(\Omega)$; see, e.g., [4,23]. As usual, the notation $H^{-s,-\sigma}(\Omega)$ is used to mean the dual space of $H^{s,\sigma}_0(\Omega)$.

2.2 Müntz Jacobi polynomials

Recall the hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(c)_i}\frac{z^i}{i!},\quad |z|<1,\quad a,b\in\mathbb{R},\quad -c\notin\mathbb{N}_0,$$

where the rising Pochhammer symbol, for $a\in\mathbb{R}$ and $i\in\mathbb{N}_0$; is defined by:

$$(a)_0=1,(a)_i=a(a+1)\cdots(a+i-1)=\frac{\Gamma(a+i)}{\Gamma(i)},\quad \text{for}\ \ i\ge 1.$$

The classical Jacobi polynomials $J_n^{\alpha,\beta}$ are defined for parameters $\alpha,\beta>-1$. The extension to $\alpha\le-1$ and/or $\beta\le-1$ was given in Szegö [36] as follows:

$$J_n^{\alpha,\beta}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\Big(-n,n+\alpha+\beta+1;\alpha+1,\frac{1-x}{2}\Big), \tag{2.13}$$

where $-n-\alpha-\beta\notin\{1,2,\cdots,n\}$.

There exist the following relationships:

$$J_n^{-l,\beta}(x)=\frac{(n-l)!(n+\beta-l+1)_l}{n!2^l}(x-1)^l J_{n-l}^{l,\beta}(x),\quad l\in\mathbb{N},\quad \beta\in\mathbb{R},\ vn\ge l\ge 1, \tag{2.14}$$

$$J_n^{\alpha,-m}(x)=\frac{(n-m)!(n+\alpha-m+1)_m}{n!2^m}(x+1)^m J_{n-m}^{\alpha,m}(x),\quad m\in\mathbb{N},\ \ \alpha\in\mathbb{R},\ \ n\ge m\ge 1. \tag{2.15}$$

Definition 2.1. The Müntz Jacobi polynomials of degree n are defined on Λ as follows:

$$^{+}J_n^{\alpha,\beta,\lambda}(x)=J_n^{\alpha,\beta}(2x^\lambda-1),\quad \forall x\in\Lambda, \tag{2.16}$$

$$^{-}J_n^{\alpha,\beta,\lambda}(x)=J_n^{\alpha,\beta}(1-2(1-x)^\lambda),\quad \forall x\in\Lambda, \tag{2.17}$$

where $J_n^{\alpha,\beta}(x)$ is defined in (2.13) and $\alpha,\beta\in\mathbb{R}$, $0<\lambda\le 1$.

When $\lambda=1$, the polynomials $\{^{+}J_n^{\alpha,\beta,1}(x)\}_{n=0}^\infty$ and $\{^{-}J_n^{\alpha,\beta,1}(x)\}_{n=0}^\infty$ are called shifted Jacobi polynomials, which are orthogonal polynomials with respect to the weight function $(1-x)^\alpha x^\beta$ for $\alpha,\beta>-1$.

Let's denote

$$^{+}\omega^{\alpha,\beta,\lambda}(x):=\lambda(1-x^\lambda)^\alpha x^{(\beta+1)\lambda-1},\quad ^{-}\omega^{\alpha,\beta,\lambda}(x):=\lambda(1-x)^{(\alpha+1)\lambda-1}\big(1-(1-x)^\lambda\big)^\beta.$$

For the special case $\lambda=1$, we have $^{+}\omega^{\alpha,\beta,1}(x)={}^{-}\omega^{\alpha,\beta,1}(x)=(1-x)^\alpha x^\beta$. In this case, we will use the notation $\omega^{\alpha,\beta,1}$ instead of $^{+}\omega^{\alpha,\beta,1}(x)$ and $^{-}\omega^{\alpha,\beta,1}(x)$.

Lemma 2.4. *For $\alpha,\beta>-1$, $0<\lambda\le1$, the Müntz Jacobi polynomials $^{+}J_n^{\alpha,\beta,\lambda}(x)$ and $^{-}J_n^{\alpha,\beta,\lambda}(x)$ are orthogonal with respect to the weight functions $^{+}\omega^{\alpha,\beta,\lambda}(x)$ and $^{-}\omega^{\alpha,\beta,\lambda}(x)$ in Λ respectively, i.e.,*

$$\int_0^1 {}^{+}\omega^{\alpha,\beta,\lambda}(x)\,{}^{+}J_m^{\alpha,\beta,\lambda}(x)\,{}^{+}J_n^{\alpha,\beta,\lambda}(x)dx=\hat{\gamma}_n^{\alpha,\beta}\delta_{m,n},$$

$$\int_0^1 {}^{-}\omega^{\alpha,\beta,\lambda}(x)\,{}^{-}J_m^{\alpha,\beta,\lambda}(x)\,{}^{-}J_n^{\alpha,\beta,\lambda}(x)dx=\hat{\gamma}_n^{\alpha,\beta}\delta_{m,n},$$

where

$$\hat{\gamma}_n^{\alpha,\beta}=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

Lemma 2.5. *For* $\alpha\in\mathbb{R}$, $0<\lambda\le 1$, *we have*

$$\int_0^x {}^+J_n^{\alpha,\frac{1}{\lambda}-1,\lambda}(s)ds=\frac{x}{\lambda(n+1/\lambda)}{}^+J_n^{\alpha-1,\frac{1}{\lambda},\lambda}(x), \tag{2.18}$$

$$\int_x^1 {}^-J_n^{\frac{1}{\lambda}-1,\alpha,\lambda}(s)ds=\frac{1-x}{\lambda(n+1/\lambda)}{}^-J_n^{\frac{1}{\lambda},\alpha-1,\lambda}(x). \tag{2.19}$$

Proof. Using suitable variable change and equality (2.9), we have

$$\begin{aligned}\int_0^x {}^+J_n^{\alpha,\frac{1}{\lambda}-1,\lambda}(s)ds&=\frac{1}{\lambda 2^{1/\lambda}}\int_{-1}^{2x^\lambda-1}(1+\xi)^{\frac{1}{\lambda}-1}J_n^{\alpha,\frac{1}{\lambda}-1}(\xi)d\xi\\ &=\frac{x}{\lambda(n+1/\lambda)}J_n^{\alpha-1,\frac{1}{\lambda}}(2x^\lambda-1)\\ &=\frac{x}{\lambda(n+1/\lambda)}{}^+J_n^{\alpha-1,\frac{1}{\lambda},\lambda}(x).\end{aligned}$$

The identity (2.19) can be proved in a similar way by using suitable variable change and (2.10). The proof is completed. □

Taking $\alpha=0$ in (2.18) and (2.19), then using (2.14) and (2.15), we obtain

$$\int_0^x {}^+J_n^{0,\frac{1}{\lambda}-1,\lambda}(s)ds=\frac{x}{\lambda(n+1/\lambda)}{}^+J_n^{-1,\frac{1}{\lambda},\lambda}(x)=-\frac{(1-x^\lambda)x}{\lambda n}{}^+J_{n-1}^{1,\frac{1}{\lambda},\lambda}(x), \tag{2.20}$$

$$\int_x^1 {}^-J_n^{\frac{1}{\lambda}-1,0,\lambda}(s)ds=\frac{1-x}{\lambda(n+1/\lambda)}{}^-J_n^{\frac{1}{\lambda},-1,\lambda}(x)=\frac{(1-(1-x)^\lambda)(1-x)}{\lambda n}{}^-J_{n-1}^{\frac{1}{\lambda},1,\lambda}(x). \tag{2.21}$$

Then we introduce the Müntz polynomial spaces as follows:

$$\begin{aligned}{}^+S_N^\lambda(\Lambda)&:=\text{span}\{{}^+\varphi_n^\lambda(x),\ n=1,2,\cdots,N\},\\ {}^-S_N^\lambda(\Lambda)&:=\text{span}\{{}^-\varphi_n^\lambda(x),\ n=1,2,\cdots,N\},\end{aligned}$$

where

$$\begin{aligned}{}^+\varphi_n^\lambda(x)&=-\frac{\sqrt{2n+1/\lambda}}{\sqrt{\lambda}n}x(1-x^\lambda){}^+J_{n-1}^{1,\frac{1}{\lambda},\lambda}(x),\\ {}^-\varphi_n^\lambda(x)&=-\frac{\sqrt{2n+1/\lambda}}{\sqrt{\lambda}n}(1-(1-x)^\lambda)(1-x){}^-J_{n-1}^{\frac{1}{\lambda},1,\lambda}(x).\end{aligned} \tag{2.22}$$

Since ${}^+\omega^{0,\frac{1}{\lambda}-1,\lambda}(x)={}^-\omega^{\frac{1}{\lambda}-1,0,\lambda}(x)=\lambda$, it follows from Lemma 2.4, (2.22), and (2.20) that both $\{\frac{d\,{}^+\varphi_n^\lambda}{dx}\}$ and $\{\frac{d\,{}^-\varphi_n^\lambda}{dx}\}$ are orthogonal set with respect to the weight function $\omega\equiv 1$.

Define the orthogonal projector ${}^+\pi_{N,\lambda}^{1,0}$: $H_0^1(\Lambda)\to {}^+S_N^\lambda(\Lambda)$, such that, for all $v\in H_0^1(\Lambda)$, ${}^+\pi_{N,\lambda}^{1,0}v\in {}^+S_N^\lambda(\Lambda)$ satisfying

$$((v-{}^+\pi_{N,\lambda}^{1,0}v)',v_N')_\Lambda=0,\quad \forall v_N\in {}^+S_N^\lambda(\Lambda), \tag{2.23}$$

and ${}^-\pi^{1,0}_{N,\lambda}$: $H^1_0(\Lambda)\to {}^-S^\lambda_N(\Lambda)$, such that, for all $v\in H^1_0(\Lambda)$, ${}^-\pi^{1,0}_{N,\lambda}v\in {}^-S^\lambda_N(\Lambda)$ satisfying

$$((v-{}^-\pi^{1,0}_{N,\lambda}v)',v'_N)_\Lambda=0,\quad \forall v_N\in {}^-S^\lambda_N(\Lambda). \tag{2.24}$$

It follows immediately from the definitions (2.23) and (2.24) that

$$|v-{}^+\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}=\inf_{v_N\in {}^+S^\lambda_N}|v-v_N|_{1,\Lambda},\quad |v-{}^-\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}=\inf_{v_N\in {}^-S^\lambda_N}|v-v_N|_{1,\Lambda}.$$

Next we will derive approximation results for the projectors ${}^+\pi^{1,0}_{N,\lambda}$ and ${}^-\pi^{1,0}_{N,\lambda}$. To this end, we introduce some more operators, and recall/study their approximation properties. We recall the orthogonal projector $\pi^{\alpha,\beta}_{N,\lambda}:L^2_{{}^+\omega^{\alpha,\beta,\lambda}}(\Lambda)\to S^{\alpha,\beta}_{N,\lambda}(\Lambda):=\mathrm{span}\{{}^+J^{\alpha,\beta,\lambda}_n(x),\, n=0,1,\cdots,N\}$ by: $\forall v\in L^2_{{}^+\omega^{\alpha,\beta,\lambda}}(\Lambda)$, $\pi^{\alpha,\beta}_{N,\lambda}v\in S^{\alpha,\beta}_{N,\lambda}(\Lambda)$, such that

$$(v-\pi^{\alpha,\beta}_{N,\lambda}v,v_N)_{{}^+\omega^{\alpha,\beta,\lambda}}=0,\quad \forall v_N\in S^{\alpha,\beta}_{N,\lambda}(\Lambda).$$

It has been proven in [14] that for all $v(\cdot)\in L^2_{{}^+\omega^{\alpha,\beta,\lambda}}(\Lambda), v(\cdot^{1/\lambda})\in B^m_{\omega^{\alpha,\beta,1}}(\Lambda)$, $m\geq 1$, it holds

$$\|v-\pi^{\alpha,\beta}_{N,\lambda}v\|_{0,{}^+\omega^{\alpha,\beta,\lambda}}\lesssim N^{-m}\|\partial^m_x v(x^{1/\lambda})\|_{0,\omega^{\alpha+m,\beta+m,1}}, \tag{2.25}$$

$$\|(v-\pi^{\alpha,\beta}_{N,\lambda}v)'\|_{0,{}^+\omega^{\alpha+1,\beta-1+1/\lambda,\lambda}}\lesssim N^{1-m}\|\partial^m_x v(x^{1/\lambda})\|_{0,\omega^{\alpha+m,\beta+m,1}}. \tag{2.26}$$

We also need a new orthogonal projector $\widehat{\pi}^{\alpha,\beta}_{N,\lambda}:L^2_{{}^-\omega^{\alpha,\beta,\lambda}}(\Lambda)\to \widehat{S}^{\alpha,\beta}_{N,\lambda}(\Lambda):=\mathrm{span}\{{}^-J^{\alpha,\beta,\lambda}_n,\, n=0,1,\cdots,N\}$, defined by: $\forall v\in L^2_{{}^-\omega^{\alpha,\beta,\lambda}}(\Lambda)$, $\widehat{\pi}^{\alpha,\beta}_{N,\lambda}v\in \widehat{S}^{\alpha,\beta}_{N,\lambda}(\Lambda)$, such that

$$(v-\widehat{\pi}^{\alpha,\beta}_{N,\lambda}v,v_N)_{{}^-\omega^{\alpha,\beta,\lambda}}=0,\quad \forall v_N\in \widehat{S}^{\alpha,\beta}_{N,\lambda}(\Lambda).$$

In a similar way as in [14], we can derive the optimal error estimates for $\widehat{\pi}^{\alpha,\beta}_{N,\lambda}v$ as follows: for all $v(\cdot)\in L^2_{{}^-\omega^{\alpha,\beta,\lambda}}, v(1-x^{1/\lambda})\in B^m_{\omega^{\alpha,\beta,1}}(\Lambda)$, $m\geq 1$,

$$\|v-\widehat{\pi}^{\alpha,\beta}_{N,\lambda}v\|_{0,{}^-\omega^{\alpha,\beta,\lambda}}\lesssim N^{-m}\|\partial^m_x v(1-x^{1/\lambda})\|_{0,\omega^{\alpha+m,\beta+m,1}},$$
$$\|(v-\widehat{\pi}^{\alpha,\beta}_{N,\lambda}v)'\|_{0,{}^-\omega^{\alpha-1+1/\lambda,\beta+1,\lambda}}\lesssim N^{1-m}\|\partial^m_x v(1-x^{1/\lambda})\|_{0,\omega^{\alpha+m,\beta+m,1}}.$$

Now we are in a position to derive error estimates for ${}^+\pi^{1,0}_{N,\lambda}v$ and ${}^-\pi^{1,0}_{N,\lambda}v$.

Theorem 2.1. *For all $v\in H^1_0(\Lambda)\cap\{v|v'(\cdot^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)\}$, $m\geq 1$, it holds*

$$|v-{}^+\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}\lesssim N^{1-m}|v'(\cdot^{1/\lambda})|_{m-1,\omega^{0,0,1}}, \tag{2.27}$$

and for all $v\in H^1_0(\Lambda)\cap\{v|v'(1-x^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)\}$, $m\geq 1$,

$$|v-{}^-\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}\lesssim N^{1-m}|v'(1-x^{1/\lambda})|_{m-1,\omega^{0,0,1}}. \tag{2.28}$$

Proof. Given $v\in H_0^1(\Lambda)$ and $(\partial_x v)(x^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)$. Let $Q=\int_0^1 \pi_{N,\lambda}^{0,1/\lambda-1}v'dx$, and set

$$v_N=\int_0^x \big(\pi_{N,\lambda}^{0,1/\lambda-1}v'-Q\big)dx.$$

Then we deduce from (2.20) that $v_N\in {}^+S_N^\lambda(\Lambda)$, $v_N'=\pi_{N,\lambda}^{0,1/\lambda-1}v'-Q$. Consequently, we have

$$\|v'-v_N'\|_{0,\Lambda}\le \|v'-\pi_{N,\lambda}^{0,1/\lambda-1}v'\|_{0,\Lambda}+|Q|. \tag{2.29}$$

Furthermore, using the Cauchy-Schwarz inequality, we obtain

$$|Q|=\Big|\int_0^1 \pi_{N,\lambda}^{0,1/\lambda-1}v'dx\Big|=\Big|\int_0^1 v'-\pi_{N,\lambda}^{0,1/\lambda-1}v'dx\Big|\le \|v'-\pi_{N,\lambda}^{0,1/\lambda-1}v'\|_{0,\Lambda}. \tag{2.30}$$

Finally, it follows from (2.25), (2.29), and (2.30):

$$\begin{aligned}|v-{}^+\pi_{N,\lambda}^{1,0}v|_{1,\Lambda}&\le |v-v_N|_{1,\Lambda}=\|v'-v_N'\|_{0,\Lambda}\le 2\|v'-\pi_{N,\lambda}^{0,1/\lambda-1}v'\|_{0,\Lambda}\\&\lesssim N^{1-m}\|\partial_x^{m-1}\{v'(\cdot^{1/\lambda})\}\|_{0,\omega^{m-1,m-1,1}}.\end{aligned}$$

This gives the error estimate (2.27) for the projector ${}^+\pi_{N,\lambda}^{1,0}$. Similarly, we can prove (2.28) for the operator ${}^-\pi_{N,\lambda}^{1,0}$. This completes the proof. $\square$

L^2-error estimates for ${}^+\pi_{N,\lambda}^{1,0}$ and ${}^-\pi_{N,\lambda}^{1,0}$ will be derived by using the Aubin-Nitsche technique, which is given in the following theorem.

Theorem 2.2. *For all* $v\in H_0^1(\Lambda)\cap\{v|\,v'(\cdot^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)\}$, $m\ge 1$, *it holds*

$$\|v-{}^+\pi_{N,\lambda}^{1,0}v\|_{0,\Lambda}\lesssim N^{-m}|v'(\cdot^{1/\lambda})|_{m-1,\omega^{0,0,1}}, \tag{2.31}$$

and for all $v\in H_0^1(\Lambda)\cap\{v|\,v'(1-x^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)\}$, $m\ge 1$,

$$\|v-{}^-\pi_{N,\lambda}^{1,0}v\|_{0,\Lambda}\lesssim N^{-m}|v'(1-x^{1/\lambda})|_{m-1,\omega^{0,0,1}}. \tag{2.32}$$

Proof. First, we consider the auxiliary problem: Given $g\in L^2_{+\omega^{-1,-1,\lambda}}(\Lambda)$, find $\Psi(g)\in H_0^1(\Lambda)$, such that

$$\mathcal{A}(\Psi(g),\varphi)=\mathcal{G}(\varphi),\quad \forall\varphi\in H_0^1(\Lambda), \tag{2.33}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by $\mathcal{A}(\cdot,\cdot):=(\cdot',\cdot')$, the functional $\mathcal{G}(\cdot)$ is defined by $\mathcal{G}(\cdot):=(g,\cdot)_{+\omega^{-1,-1,\lambda}}$.

It is obvious that $\mathcal{A}(\cdot,\cdot)$ is continuous and coercive in $H_0^1(\Lambda)\times H_0^1(\Lambda)$. On the other side, using Lemma 4.7 in [15], we have

$$|\mathcal{G}(\varphi)|\le \|g\|_{0,+\omega^{-1,-1,\lambda}}\|\varphi\|_{0,+\omega^{-1,-1,\lambda}}\lesssim \|g\|_{0,+\omega^{-1,-1,\lambda}}\|\varphi'\|_{0,+\omega^{0,2/\lambda-2,\lambda}}\lesssim \|g\|_{0,+\omega^{-1,-1,\lambda}}|\varphi|_{1,\Lambda}.$$

Thus the linear functional $\mathcal{G}(\cdot)\in H^{-1}(\Lambda)$. Then it follows from the Lax-Milgram lemma that the problem (2.33) admits a unique solution $\Psi(g)$. Moreover, in the distribution sense, we have $\Psi(g)''=-{}^{+}\omega^{-1,-1,\lambda}g$. Consequently, a direct calculation gives

$$\begin{aligned}\|g\|_{0,{}^{+}\omega^{-1,-1,\lambda}}&=\Big(\int_0^1\frac{1}{\lambda}\big(\Psi(g)''(x)\big)^2(1-x^{\lambda})xdx\Big)^{1/2}\\&=\Big(\int_0^1\frac{1}{\lambda}\Big(\frac{d}{dy^{1/\lambda}}\{\Psi(g)'(y^{1/\lambda})\}\Big)^2(1-y)y^{1/\lambda}dy^{1/\lambda}\Big)^{1/2}\\&=\sqrt{\lambda}\|\partial_x\{\Psi(g)'(x^{1/\lambda})\}\|_{0,\omega^{1,1,1}}.\end{aligned}$$

Now given $v\in H_0^1(\Lambda)\cap\{v|\,v'(\cdot^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda)\}$, $m\ge 1$, using (2.23) and (2.27), and taking $\varphi=v-{}^{+}\pi^{1,0}_{N,\lambda}v$ in (2.33), we get

$$\begin{aligned}(v-{}^{+}\pi^{1,0}_{N,\lambda}v,g)_{{}^{+}\omega^{-1,-1,\lambda}}&=\big((v-{}^{+}\pi^{1,0}_{N,\lambda}v)',\Psi(g)'\big)=\big((v-{}^{+}\pi^{1,0}_{N,\lambda}v)',(\Psi(g)-{}^{+}\pi^{1,0}_{N,\lambda}\Psi(g))'\big)\\&\le|v-{}^{+}\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}|\Psi(g)-{}^{+}\pi^{1,0}_{N,\lambda}\Psi(g)|_{1,\Lambda}\\&\lesssim N^{-1}|v-{}^{+}\pi^{1,0}_{N,\lambda}v|_{1,\Lambda}\|\partial_x\{\Psi(g)'(x^{1/\lambda})\}\|_{0,\omega^{1,1,1}}\\&\lesssim N^{-m}|v'(\cdot^{1/\lambda})|_{m-1,\omega^{0,0,1}}\|g\|_{0,{}^{+}\omega^{-1,-1,\lambda}}.\end{aligned}$$

Thus

$$\begin{aligned}\|v-{}^{+}\pi^{1,0}_{N,\lambda}v\|_{0,\Lambda}\le\|v-{}^{+}\pi^{1,0}_{N,\lambda}v\|_{0,{}^{+}\omega^{-1,-1,\lambda}}&=\sup_{0\neq g\in L^2_{{}^{+}\omega^{-1,-1,\lambda}}(\Lambda)}\frac{(v-{}^{+}\pi^{1,0}_{N,\lambda}v,g)_{{}^{+}\omega^{-1,-1,\lambda}}}{\|g\|_{0,{}^{+}\omega^{-1,-1,\lambda},\Lambda}}\\&\lesssim N^{-m}|v'(\cdot^{1/\lambda})|_{m-1,\omega^{0,0,1}},\end{aligned}$$

which gives the desired estimate (2.31). We can derive (2.32) for ${}^{-}\pi^{1,0}_{N,\lambda}$ in a similar way. This completes the proof. □

In virtue of usual interpolation theorem, the following estimates hold for all $0\le s\le 1$:

$$\|v-{}^{+}\pi^{1,0}_{N,\lambda}v\|_{s,\Lambda}\lesssim N^{s-m}|v'(\cdot^{1/\lambda})|_{m-1,\omega^{0,0,1}},\ \ \forall v'(\cdot^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda),\tag{2.34}$$

$$\|v-{}^{-}\pi^{1,0}_{N,\lambda}v\|_{s,\Lambda}\lesssim N^{s-m}|v'(1-x^{1/\lambda})|_{m-1,\omega^{0,0,1}},\ \ \forall v'(1-x^{1/\lambda})\in B^{m-1}_{\omega^{0,0,1}}(\Lambda).\tag{2.35}$$

In the next lemma, we study properties of the composite approximation operator ${}^{+}\pi^{1,0}_{N_1,\lambda_1}{}^{+}\pi^{1,0}_{N_2,\lambda_2}$.

Lemma 2.6. *For all* $v\in H_0^1(\Omega)$,

$$\begin{aligned}&\partial_x v(x^{1/\lambda},y)\in B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda))\cap B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;H^{\sigma}(\Lambda)),\\&\partial_y v(x,y^{1/\lambda})\in L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))\cap H^s(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)),\end{aligned}$$

and

$$\partial_{xy}v(x^{1/\lambda},y^{1/\lambda})\in B^{m_1-1}_{\omega^{0,0,1}}\big(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)\big),\quad 0\le s,\sigma\le 1,$$

we have

$$\begin{aligned}\|v-{}^+\pi^{1,0}_{N_1,\lambda_1}{}^+\pi^{1,0}_{N_2,\lambda_2}v\|_{s,0}\lesssim & N_1^{s-m_1}\|\partial_x v(x^{1/\lambda_1},y)\|_{B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda))}\\ &+N_1^{s-m_1}N_2^{-m_2}\|\partial_{xy}v(x^{1/\lambda_1},y^{1/\lambda_2})\|_{B^{m_1-1}_{\omega^{0,0,1}}\left(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)\right)}\\ &+N_2^{-m_2}\|\partial_y v(x,y^{1/\lambda_2})\|_{H^s(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))},\end{aligned}$$

and

$$\begin{aligned}\|v-{}^+\pi^{1,0}_{N_1,\lambda_1}{}^+\pi^{1,0}_{N_2,\lambda_2}v\|_{0,\sigma}\lesssim & N_2^{\sigma-m_2}\|\partial_y v(x,y^{1/\lambda_2})\|_{L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))}\\ &+N_1^{-m_1}N_2^{\sigma-m_2}\|\partial_{xy}v(x^{1/\lambda_1},y^{1/\lambda_2})\|_{B^{m_1-1}_{\omega^{0,0,1}}\left(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)\right)}\\ &+N_1^{-m_1}\|\partial_x v(x^{1/\lambda_1},y)\|_{B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;H^\sigma(\Lambda))}.\end{aligned}$$

Proof. Using the triangular inequality and (2.34), we have

$$\begin{aligned}&\|v-{}^+\pi^{1,0}_{N_1,\lambda_1}{}^+\pi^{1,0}_{N_2,\lambda_2}v\|_{s,0}\\ \le&\|v-{}^+\pi^{1,0}_{N_1,\lambda_1}v\|_{s,0}+\|({}^+\pi^{1,0}_{N_1,\lambda_1}-I_d)(v-{}^+\pi^{1,0}_{N_2,\lambda_2}v)\|_{s,0}+\|v-{}^+\pi^{1,0}_{N_2,\lambda_2}v)\|_{s,0}\\ \lesssim& N_1^{s-m_1}\|\partial_x v(x^{1/\lambda_1},y)\|_{B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda))}+N_1^{s-m_1}N_2^{-m_2}\|\partial_{xy}v(x^{1/\lambda_1},y^{1/\lambda_2})\|_{B^{m_1-1}_{\omega^{0,0,1}}\left(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)\right)}\\ &+N_2^{-m_2}\|\partial_y v(x^{1/\lambda_1},y)\|_{H^s(\Lambda;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda))}.\end{aligned}$$

In the same way, we can obtain the second estimate. This completes the proof. ☐

3 A second order time stepping scheme

It is convenient to introduce the semi-norm $|\cdot|_{1,\Omega}$, defined by

$$|v|_{1,\Omega}:=\big[\nu(\nabla v,\nabla v)_\Omega+p({}_0D_x^{\mu_1}v,\ {}_xD_1^{\mu_1}v)_\Omega+q({}_0D_y^{\mu_2}v,\ {}_yD_1^{\mu_2}v)_\Omega\big]^{1/2},\ \forall v\in H_0^1(\Omega),\ \nu>0, \tag{3.1}$$

and the semi-norm $|\cdot|_{H_0^{\mu_1,\mu_2}}$ by

$$|v|_{H_0^{\mu_1,\mu_2}}:=\big[p({}_0D_x^{\mu_1}v,{}_xD_1^{\mu_1}v)_\Omega+q({}_0D_y^{\mu_2}v,\ {}_yD_1^{\mu_2}v)_\Omega\big]^{1/2},\quad\forall v\in H_0^{\mu_1,\mu_2}(\Omega),\ 0\le\mu_1,\mu_2<1,$$

which are respectively equivalent to the $H^{1,1}$-norm and H^{μ_1,μ_2}-norm defined in (2.12).

We begin with constructing the time stepping scheme. Let $t_k=k\tau$, $k=0,1,\cdots,K$, where $\tau:=\frac{T}{K}$ is the time step size. Denote $t_{k+1/2}:=\frac{t_k+t_{k+1}}{2}$.

Consider Eq. (2.1) at time $t_{k+1/2}$, and apply the Taylor formula to the different terms, e.g,

$$\begin{aligned}\frac{\partial u(x,y,t_{k+1/2})}{\partial t}&=\frac{u(x,y,t_{k+1})-u(x,y,t_k)}{\tau}-R_1^k(x,y,\tau),\\ u(x,y,t_{k+1/2})&=\frac{u(x,y,t_{k+1})+u(x,y,t_k)}{2}-R_2^k(x,y,\tau),\end{aligned}$$

where $R_1^k(x,y,\tau)$ and $R_2^k(x,y,\tau)$ are the truncation errors:

$$\begin{aligned}R_1^k(x,y,\tau)&=\frac{1}{2\tau}\Big[\int_{t_k}^{t_{k+1/2}}(t-t_k)^2\partial_{ttt}u(x,y,t)dt+\int_{t_{k+1/2}}^{t_{k+1}}(t_{k+1}-t)^2\partial_{ttt}u(x,y,t)dt\Big],\\ R_2^k(x,y,\tau)&=\frac{1}{2}\Big[\int_{t_k}^{t_{k+1/2}}(t-t_k)\partial_{tt}u(x,y,t)dt+\int_{t_{k+1/2}}^{t_{k+1}}(t_{k+1}-t)\partial_{tt}u(x,y,t)dt\Big].\end{aligned}\tag{3.2}$$

Then Eq. (2.1) at time $t_{k+1/2}$ becomes:

$$\begin{aligned}&\frac{u(x,y,t_{k+1})-u(x,y,t_k)}{\tau}+(-\nu\Delta+p\,{}_0D_x^{2\mu_1}+q\,{}_0D_y^{2\mu_2})\frac{u(x,y,t_{k+1})+u(x,y,t_k)}{2}\\ &=f(x,y,t_{k+1/2})+R^k(x,y,\tau),\end{aligned}\tag{3.3}$$

where $R^k(x,y,\tau)=R_1^k(x,y,\tau)+(-\nu\Delta+p\,{}_0D_x^{2\mu_1}+q\,{}_0D_y^{2\mu_2})R_2^k(x,y,\tau)$.

We assume

$$\begin{aligned}&\partial_{tt}u\in L^2(I;H_0^1(\Omega)),\quad \partial_{ttt}u\in L^2(I;H^{-1}(\Omega)),\quad \text{if } \nu>0;\\ &\partial_{tt}u\in L^2(I;H_0^{\mu_1,\mu_2}(\Omega)),\quad \partial_{ttt}u\in L^2(I;H^{-\mu_1,-\mu_2}(\Omega)),\quad \text{if } \nu=0.\end{aligned}\tag{3.4}$$

Then it is not difficult to verify that the truncation errors have the following upper bound:

$$\sum_{k=0}^{[T/\tau]}\tau\|R^k(x,y,\tau)\|_{-1}^2\le\tau^4\int_0^T[\|\partial_{ttt}u\|_{-1}^2+\|\partial_{tt}u\|_1^2]dt,\quad \text{if } \nu>0,\tag{3.5}$$

and

$$\sum_{k=0}^{[T/\tau]}\tau\|R^k(x,y,\tau)\|_{H^{-\mu_1,-\mu_2}(\Omega)}^2\le\tau^4\int_0^T[\|\partial_{ttt}u\|_{H^{-\mu_1,-\mu_2}(\Omega)}^2+\|\partial_{tt}u\|_{H_0^{\mu_1,\mu_2}(\Omega)}^2]dt,\quad \text{if } \nu=0.\tag{3.6}$$

The expression (3.3) suggests the following scheme:

$$\frac{u^{k+1}-u^k}{\tau}+(-\nu\Delta+p\,{}_0D_x^{2\mu_1}+q\,{}_0D_y^{2\mu_2})\frac{u^{k+1}+u^k}{2}=f^{k+1/2},\tag{3.7}$$

where $u^{k+1}(\cdot)$ is an approximation to $u(\cdot,t_{k+1})$.

Formally, the estimates (3.5) and (3.6) imply that (3.7) is a second order scheme. A rigorous proof of this fact will be given later after the following stability analysis.

Theorem 3.1. *The semi-discrete problem* (3.7) *is unconditionally stable in the sense that for all* $\tau>0$, $k=1,2,\cdots,K$, *it holds*

$$\|u^{k+1}\|_{0,\Omega}\lesssim\|u^0\|_{0,\Omega}+\sqrt{\frac{T}{2}}\|f\|_{L^\infty(I;H^{-1}(\Omega))},\quad \text{if } \nu>0, \tag{3.8}$$

and

$$\|u^{k+1}\|_{0,\Omega}\lesssim\|u^0\|_{0,\Omega}+\sqrt{\frac{T}{2}}\|f\|_{L^\infty(I;H^{-\mu_1,-\mu_2}(\Omega))},\quad \text{if } \nu=0. \tag{3.9}$$

Proof. Multiplying both sides of (3.7) by $\tau(u^{k+1}+u^k)$ and integrating the resulting equation over Ω, we obtain

$$\begin{aligned}\|u^{k+1}\|^2_{0,\Omega}-\|u^k\|^2_{0,\Omega}&+\frac{\tau\nu}{2}\|\nabla(u^{k+1}+u^k)\|^2_{0,\Omega}+\frac{\tau p}{2}({}_0D_x^{\mu_1}(u^{k+1}+u^k),{}_xD_1^{\mu_1}(u^{k+1}+u^k))_\Omega\\&+\frac{\tau q}{2}({}_0D_y^{\mu_2}(u^{k+1}+u^k),{}_yD_1^{\mu_2}(u^{k+1}+u^k))_\Omega=\tau(f^{k+1/2},u^{k+1}+u^k)_\Omega.\end{aligned} \tag{3.10}$$

In the case $\nu>0$, it follows from (3.1) and (3.10) that

$$\|u^{k+1}\|^2_{0,\Omega}-\|u^k\|^2_{0,\Omega}+\frac{\tau}{2}|u^{k+1}+u^k|^2_{1,\Omega}=\tau(f^{k+1/2},u^{k+1}+u^k)_\Omega. \tag{3.11}$$

The right hand side can be bounded by using the Poincaré inequality and Young's inequality as follows:

$$\begin{aligned}\tau(f^{k+1/2},u^{k+1}+u^k)_\Omega&\le\tau\|f^{k+1/2}\|_{-1,\Omega}\|u^{k+1}+u^k\|_{1,\Omega}\lesssim\tau\|f^{k+1/2}\|_{-1,\Omega}|u^{k+1}+u^k|_{1,\Omega}\\&\le c\frac{\tau}{2}\|f^{k+1/2}\|^2_{-1,\Omega}+\frac{\tau}{2}|u^{k+1}+u^k|^2_{1,\Omega}.\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) gives

$$\|u^{k+1}\|^2_{0,\Omega}-\|u^k\|^2_{0,\Omega}\lesssim\frac{\tau}{2}\|f^{k+1/2}\|^2_{-1,\Omega}.$$

Summing up for k from 0 to K, we obtain

$$\|u^{k+1}\|^2_{0,\Omega}\lesssim\|u^0\|^2_{0,\Omega}+\frac{T}{2}\|f\|^2_{L^\infty(I;H^{-1}(\Omega))}.$$

This proves (3.8).

In the case $\nu=0$, we can prove (3.9) in the same manner. $\square$

Theorem 3.2. *Let* u *be the exact solution of* (2.1) *and* $\{u^k\}_{k=0}^K$ *be the solution of* (3.7) *with the initial condition* $u^0(x,y)=u_0(x,y)$. *If* $\partial_{tt}u$ *and* $\partial_{ttt}u$ *satisfy the assumption* (3.4), *then we have the following error estimate:*

$$\begin{aligned}&\Big(\|u(t_k)-u^k\|^2_{0,\Omega}+\tau\sum_{n=0}^{k}\Big|\frac{u(t_{n+1})+u(t_n)}{2}-\frac{u^{n+1}+u^n}{2}\Big|^2_{1,\Omega}\Big)^{1/2}\\&\lesssim\big(\|\partial_{tt}u\|_{L^2(I;H^1_0(\Omega))}+\|\partial_{ttt}u\|_{L^2(I;H^{-1}(\Omega))}\big)\tau^2,\quad \text{if } \nu>0;\end{aligned} \tag{3.13}$$

and

$$\Big(\|u(t_k)-u^k\|_{0,\Omega}^2+\tau\sum_{n=0}^{k}\Big|\frac{u(t_{n+1})+u(t_n)}{2}-\frac{u^{n+1}+u^n}{2}\Big|^2_{H_0^{\mu_1,\mu_2}}\Big)^{1/2}$$
$$\lesssim\big(\|\partial_{tt}u\|_{L^2(I;H_0^{\mu_1,\mu_2}(\Omega))}+\|\partial_{ttt}u\|_{L^2(I;H^{-\mu_1,-\mu_2}(\Omega))}\big)\tau^2,\quad \text{if } \nu=0. \tag{3.14}$$

Proof. We only prove the estimate (3.13) for the case $\nu>0$. Estimate (3.14) for the case $\nu=0$ can be derived similarly. Let $e^k:=u(t_k)-u^k$. We deduce from (3.3) and (3.7):

$$\Big(\frac{e^{k+1}-e^k}{\tau},v\Big)_\Omega+\frac{1}{2}\big[\nu(\nabla(e^{k+1}+e^k),\nabla v)_\Omega+p({}_0D_x^{\mu_1}(e^{k+1}+e^k),{}_xD_1^{\mu_1}v)_\Omega$$
$$+q({}_0D_y^{\mu_2}(e^{k+1}+e^k),{}_yD_1^{\mu_1}v)_\Omega\big]=(R^k(x,y,\tau),v)_\Omega,\quad \forall v\in H_0^1(\Omega). \tag{3.15}$$

Taking $v=2\tau e^{k+1/2}:=2\tau\frac{e^{k+1}+e^k}{2}$ in (3.15), using the Young's inequality, Lemma 2.2, and Lemma 2.3, we obtain

$$\|e^{k+1}\|_{0,\Omega}^2-\|e^k\|_{0,\Omega}^2+2\tau|e^{k+1/2}|_{1,\Omega}^2\le c\tau\|R^k(x,y,\tau)\|_{-1,\Omega}^2+\tau|e^{k+1/2}|_{1,\Omega}^2.$$

Then summing up the above inequality over k and applying the estimate (3.5), we get

$$\|e^{k+1}\|_{0,\Omega}^2+\tau\sum_{n=0}^{k}|e^{n+1/2}|_{1,\Omega}^2\lesssim\big(\|\partial_{tt}u\|^2_{L^2(I;H_0^1(\Omega))}+\|\partial_{ttt}u\|^2_{L^2(I;H^{-1}(\Omega))}\big)\tau^4.$$

Hence we complete the proof for (3.13). □

4 Müntz spectral method for the spatial discretization

Now we turn to construct the Müntz spectral method for the discretization in space of (3.7). The proposed method consists in finding $u_{N_1,N_2}^{k+1}\in S_{N_1}^{\lambda_1}(\Lambda)\otimes S_{N_2}^{\lambda_2}(\Lambda)$, such that

$$\Big(\frac{u_{N_1,N_2}^{k+1}-u_{N_1,N_2}^k}{\tau},v_{N_1,N_2}\Big)+\frac{1}{2}\big[\nu(\nabla(u_{N_1,N_2}^{k+1}+u_{N_1,N_2}^k),\nabla v_{N_1,N_2})$$
$$+p({}_0D_x^{\mu_1}(u_{N_1,N_2}^{k+1}+u_{N_1,N_2}^k),{}_xD_1^{\mu_1}v_{N_1,N_2})+q({}_0D_y^{\mu_2}(u_{N_1,N_2}^{k+1}+u_{N_1,N_2}^k),{}_yD_1^{\mu_2}v_{N_1,N_2})\big] \tag{4.1}$$
$$=(f^{k+1/2},v_{N_1,N_2}),\quad \forall v_{N_1,N_2}\in S_{N_1}^{\lambda_1}(\Lambda)\otimes S_{N_2}^{\lambda_2}(\Lambda). \tag{4.2}$$

For u_{N_1,N_2}^k given, the well-posedness of the problem (4.1) is guaranteed by the well-known Lax-Milgram lemma; see Li and Xu [19,20].

Here we are interested in deriving an error estimate for the full discrete solution $\{u_{N_1,N_2}^k\}_{k=0}^K$. For the sake of simplification, we denote $N:=(N_1,N_2)$, $u_N^k:=u_{N_1,N_2}^k$.

Additional to the assumption (3.4), we suppose that the exact solution u satisfies the following conditions:

$$\begin{aligned}&\{\partial_{xt}u\}(x^{1/\lambda_1},y,t)\in L^2(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda))),\quad \{\partial_{yt}u\}(x,y^{1/\lambda_2},t)\in L^2(I;L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))),\\&\{\partial_{yxt}u\}(x^{1/\lambda_1},y^{1/\lambda_2},t)\in L^2(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))),\\&\{\partial_{x}u\}(x^{1/\lambda_1},y,t)\in C(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;H^{\sigma}(\Lambda))),\quad \{\partial_{y}u\}(x,y^{1/\lambda_2},t)\in C(I;L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))),\\&\{\partial_{yx}u\}(x^{1/\lambda_1},y^{1/\lambda_2},t)\in C(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda))).\end{aligned}\tag{4.3}$$

The main error estimate is given in the following lemma.

Theorem 4.1. *Let u be the exact solution of problem* (2.1) *and $\{u_N^k\}$ be the solution of the full discrete problem* (4.1) *with the initial condition u_N^0 taken to $\pi^{1,0}_{N_1,\lambda_1}\pi^{1,0}_{N_2,\lambda_2}u_0$. If the exact solution u satisfies the assumptions* (3.4) *and* (4.3), *then we have*

$$\begin{aligned}&\left[\|u(\cdot,t_{k+1})-u_N^{k+1}\|^2_{0,\Omega}+\sum_{n=0}^{k}\left\|\frac{u(\cdot,t_{n+1})-u_N^{n+1}}{2}-\frac{u(\cdot,t_n)-u_N^n}{2}\right\|^2_{H^{s,\sigma}}\right]^{1/2}\\&\lesssim\tau^2+N_1^{-m_1}\|\partial_{xt}u(x^{1/\lambda_1},y,t)\|_{L^2(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda)))}\\&\quad+N_1^{-m_1}N_2^{-m_2}\|\partial_{xyt}u(x^{1/\lambda_1},y^{1/\lambda_2},t)\|_{L^2(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)))}\\&\quad+N_2^{-m_2}\|\partial_{yt}u(x,y^{1/\lambda_2},t)\|_{L^2(I;L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)))}\\&\quad+N_1^{s-m_1}\|\partial_{x}u(x^{1/\lambda_1},y,t)\|_{C(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda)))}+N_2^{-m_2}\|\partial_{x}u(x,y^{1/\lambda_2},t)\|_{C(I;H^s(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)))}\\&\quad+N_1^{s-m_1}N_2^{\sigma-m_2}\|\partial_{xy}u(x^{1/\lambda_1},y^{1/\lambda_2},t)\|_{C(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)))}\\&\quad+N_2^{\sigma-m_2}\|\partial_{y}u(x,y^{1/\lambda_2},t)\|_{C(I;L^2(\Lambda;B^{m_2-1}_{\omega^{0,0,1}}(\Lambda)))}\\&\quad+N_1^{-m_1}\|\partial_{x}u(x^{1/\lambda_1},y,t)\|_{C(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;H^{\sigma}(\Lambda)))},\end{aligned}\tag{4.4}$$

where $(s,\sigma)=(1,1)$ if $\nu>0$, and $(s,\sigma)=(\mu_1,\mu_2)$ if $\nu=0$.

Proof. Let

$$\begin{aligned}&\widehat{e}_N^k=u(\cdot,t_k)-{}^+\pi^{1,0}_{N_1,\lambda_1}{}^+\pi^{1,0}_{N_2,\lambda_2}u(\cdot,t_k),\\&\widetilde{e}_N^k={}^+\pi^{1,0}_{N_1,\lambda_1}{}^+\pi^{1,0}_{N_2,\lambda_2}u(\cdot,t_k)-u_N^k,\\&\widehat{e}_N^{k+1/2}=\frac{\widehat{e}_N^{k+1}+\widehat{e}_N^k}{2},\quad \widetilde{e}_N^{k+1/2}=\frac{\widetilde{e}_N^{k+1}+\widetilde{e}_N^k}{2}.\end{aligned}$$

First, it follows from (4.1) and (3.3):

$$\begin{aligned}&\Big(\frac{\widetilde{e}_N^{k+1}-\widetilde{e}_N^k}{\tau},v_N\Big)+\big[\nu(\nabla\widetilde{e}_N^{k+1/2},\nabla v_N)+p({}_0D_x^{\mu_1/2}(\widetilde{e}_N^{k+1/2}),{}_xD_1^{\mu_1/2}v_N)\\&\qquad+q({}_0D_y^{\mu_2/2}\widetilde{e}_N^{k+1/2},{}_yD_1^{\mu_2/2}v_N)\big]\\&=(R^k,v_N)+\frac{1}{\tau}\big(({}^+\pi^{1,0}_{N_1,\lambda_1}\,{}^+\pi^{1,0}_{N_2,\lambda_2}-I)(u(t_{k+1})-u(t_k)),v_N\big)\\&\quad-\big[\nu(\nabla\widehat{e}_N^{k+1/2},\nabla v_N)+p({}_0D_x^{\mu_1/2}\widehat{e}_N^{k+1/2},{}_xD_1^{\mu_1/2}v_N)+q({}_0D_y^{\mu_2/2}\widehat{e}_N^{k+1/2},{}_yD_1^{\mu_2/2}v_N)\big].\end{aligned}\tag{4.5}$$

In the case $\nu>0$, we deduce from taking $v_N=2\tau\widetilde{e}_N^{k+1/2}$, then using the Young's inequality, Lemma 2.2, and Lemma 2.3:

$$\begin{aligned}\|\widetilde{e}_N^{k+1}\|^2_{0,\Omega}-\|\widetilde{e}_N^{k}\|^2_{0,\Omega}+2\tau|\widetilde{e}_N^{k+1/2}|^2_{1,\Omega}\le{}&c_0\tau\|R^k\|_{-1,\Omega}+\frac{\tau}{2}|\widetilde{e}_N^{k+1/2}|_{1,\Omega}\\&+\int_{t_k}^{t_{k+1}}\|({}^+\pi^{1,0}_{N_1,\lambda_1}\,{}^+\pi^{1,0}_{N_2,\lambda_2}-I)\partial_t u\|^2_{0,\Omega}dt\\&+\tau\|\widetilde{e}_N^{k+1/2}\|^2_{0,\Omega}+c_1\tau|\widehat{e}_N^{k+1/2}|^2_{1,\Omega}+\frac{\tau}{2}|\widetilde{e}_N^{k+1/2}|^2_{1,\Omega}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\widetilde{e}_N^{k+1}\|^2_{0,\Omega}-\|\widetilde{e}_N^{k}\|^2_{0,\Omega}+\tau|\widetilde{e}_N^{k+1/2}|^2_{1,\Omega}\le{}&c_0\tau\|R^k\|_{-1,\Omega}+\int_{t_k}^{t_{k+1}}\|({}^+\pi^{1,0}_{N_1,\lambda_1}\,{}^+\pi^{1,0}_{N_2,\lambda_2}-I)\partial_t u\|^2_{0,\Omega}dt\\&+\tau\|\widetilde{e}_N^{k+1/2}\|^2_{0,\Omega}+c_1\tau|\widehat{e}_N^{k+1/2}|^2_{1,\Omega}\\\le{}&c_0\tau\|R^k\|_{-1,\Omega}+\int_{t_k}^{t_{k+1}}\|({}^+\pi^{1,0}_{N_1,\lambda_1}\,{}^+\pi^{1,0}_{N_2,\lambda_2}-I)\partial_t u\|^2_{0,\Omega}dt\\&+\frac{\tau}{2}\big(\|\widetilde{e}_N^{k+1}\|^2_{0,\Omega}+\|\widetilde{e}_N^{k}\|^2_{0,\Omega}\big)+\frac{c_1\tau}{2}\big(|\widehat{e}_N^{k+1}|^2_{1,\Omega}+|\widehat{e}_N^{k}|^2_{1,\Omega}\big).\end{aligned}$$

Summing up the above inequality over k and using (3.5), we obtain

$$\begin{aligned}&\|\widetilde{e}_N^{k+1}\|^2_{0,\Omega}-\|\widetilde{e}_N^{0}\|^2_{0,\Omega}+\tau\sum_{n=0}^{k}|\widetilde{e}_N^{n+1/2}|^2_{1,\Omega}\\\le{}&c_0\tau^4\int_0^T\big(\|\partial_{ttt}u\|^2_{-1}+\|\partial_{tt}u\|^2_1\big)dt+\|({}^+\pi^{1,0}_{N_1,\lambda_1}\,{}^+\pi^{1,0}_{N_2,\lambda_2}-I)\partial_t u\|^2_{L^2(I;L^2(\Omega))}\\&+\frac{\tau}{2}\sum_{n=0}^{k}\big(\|\widetilde{e}_N^{n+1}\|^2_{0,\Omega}+\|\widetilde{e}_N^{n}\|^2_{0,\Omega}\big)+\frac{c_1\tau}{2}\sum_{n=0}^{k}\big(|\widehat{e}_N^{n+1}|^2_{1,\Omega}+|\widehat{e}_N^{n}|^2_{1,\Omega}\big).\end{aligned}$$

Applying the discrete Gronwall lemma and noticing $\widetilde{e}^0=0$, we get

$$
\begin{aligned}
&\|\widetilde{e}_N^{k+1}\|_{0,\Omega}^2+\tau\sum_{n=0}^{k}|\widetilde{e}_N^{n+1/2}|_{1,\Omega}^2\\
\lesssim&\tau^4\int_0^T[\|\partial_{ttt}u\|_{-1}^2+\|\partial_{tt}u\|_1^2]dt\\
&+\|({}^+\pi_{N_1,\lambda_1}^{1,0}\,{}^+\pi_{N_2,\lambda_2}^{1,0}-I)\partial_t u\|_{L^2(I;L^2(\Omega))}^2+c_1\|({}^+\pi_{N_1,\lambda_1}^{1,0}\,{}^+\pi_{N_2,\lambda_2}^{1,0}-I)u\|_{C(I;H^1(\Omega))}^2\big].
\end{aligned}
$$

Let $e_N^k=u(\cdot,t_k)-u_N^k$, which is indeed equal to $\widehat{e}_N^k+\widetilde{e}_N^k$. Then the estimate (4.4) for $\nu>0$ follows from the triangle inequality

$$
\|e_N^{k+1}\|_m\le\|\widetilde{e}_N^{k+1}\|_m+\|\widehat{e}_N^{k+1}\|_m,\quad m=0,1
$$

and Lemma 2.3 with $s=\sigma=1$.

The case $\nu=0$ can be done in a similar way. This completes the proof. $\square$

Remark 4.1. It is observed from Theorem 4.1 that there are several sources of the space error. However it is worthy to emphasize that high order decay rate of all the contributed error sources can be guaranteed as long as the exact solution has high regularity after suitable variable change in each direction. For example, if we take a closer look at the first error source, i.e., $N_1^{-m_1}\big\|\partial_{xt}u(x^{1/\lambda_1},y,t)\big\|_{L^2(I;B^{m_1-1}_{\omega^{0,0,1}}(\Lambda;L^2(\Lambda)))}$, we see that the error decays with a rate, which only depends on the regularity of the transformed solution in x-direction, and exponential convergence would be achieved if $\partial_{xt}u(x^{1/\lambda_1},y,t)$ is smooth with respect to the x-variable. This is a very desirable characteristic of the new method since it has been known that the variable change like $x\to x^{1/\lambda}$ with suitable λ can make the exact solution of a large class of FDEs high regular.

5 Implementation of the method

Besides the theoretical result obtained in the previous section, we want to show in this section that the proposed method can be efficiently implemented by choosing suitable basis functions.

We express the numerical solution by

$$
u_N^k=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\hat{u}_{ij}^k\,{}^+\varphi_i^{\lambda_1}(x)\,{}^+\varphi_j^{\lambda_2}(y),
$$

where ${}^+\varphi_i^{\lambda}$ are the basis functions defined in (2.22), $U^k:=(\hat{u}_{ij}^k)_{i,j=1}^{N_1,N_2}$ is the unknown tensorial vector. Then taking $\{{}^+\varphi_i^{\lambda}\}_{i,j=1}^{N_1,N_2}$ as the test functions in (4.1), we arrive at following linear system:

$$M_{\lambda_1}\big(U^{k+1}-U^k\big)M_{\lambda_2}+\frac{\tau}{2}\Big\{\nu\big[\big(U^{k+1}+U^k\big)M_{\lambda_2}+M_{\lambda_1}\big(U^{k+1}+U^k\big)\big] \\ +pS_{\lambda_1}^{\mu_1}\big(U^{k+1}+U^k\big)M_{\lambda_2}+qM_{\lambda_1}\big(U^{k+1}+U^k\big)\big(S_{\lambda_2}^{\mu_2}\big)^T\Big\}=F^{k+1}, \tag{5.1}$$

where M_λ is the mass matrix:

$$\begin{aligned}\big(M_\lambda\big)_{i,j}=&({}^+\varphi_j^\lambda,{}^+\varphi_i^\lambda)_\Lambda\\ =&\frac{\sqrt{(2i+1/\lambda)(2j+1/\lambda)}}{ij\lambda}\big((1-x^\lambda)xJ_{j-1}^{1,1/\lambda}(2x^\lambda-1),(1-x^\lambda)xJ_{i-1}^{1,1/\lambda}(2x^\lambda-1)\big)_\Lambda\\ =&\frac{\sqrt{(2i+1/\lambda)(2j+1/\lambda)}}{ij\lambda^2}\int_0^1(1-x)^2x^{3/\lambda-1}J_{j-1}^{1,1/\lambda}(2x-1)J_{i-1}^{1,1/\lambda}(2x-1)dx\\ =&\frac{\sqrt{(2i+1/\lambda)(2j+1/\lambda)}}{ij\lambda^2}\sum_{k=0}^N\rho_kJ_{i-1}^{1,1/\lambda}(2\xi_k-1)J_{j-1}^{1,1/\lambda}(2\xi_k-1),\end{aligned}$$

S_λ^μ is the fractional stiffness matrix:

$$\begin{aligned}\big(S_\lambda^\mu\big)_{i,j}=&\big({}_0D_x^{\mu\,+}\varphi_j^\lambda,{}_xD_1^{\mu\,+}\varphi_i^\lambda\big)_\Lambda=-\big({}_0I_x^{2-2\mu}\partial_x{}^+\varphi_j^\lambda,\partial_x{}^+\varphi_i^\lambda\big)_\Lambda\\ =&-\lambda\sqrt{(2i+1/\lambda)(2j+1/\lambda)}\big({}_0I_x^{2-2\mu}J_j^{0,1/\lambda-1}(2x^\lambda-1),J_i^{0,1/\lambda-1}(2x^\lambda-1)\big)_\Lambda\\ =&-\frac{\sqrt{(2i+1/\lambda)(2j+1/\lambda)}}{\lambda\Gamma(2-2\mu)}\int_0^1\int_0^1(1-\sigma)^{1-2\mu}\sigma^{1/\lambda-1}\Big(\frac{1-\sigma^{1/\lambda}}{1-\sigma}\Big)^{1-2\mu}\\ &\qquad J_j^{0,1/\lambda-1}(2\sigma x-1)d\sigma x^{\frac{3-2\mu}{\lambda}-1}J_i^{0,1/\lambda-1}(2x-1)dx\\ =&-\frac{\sqrt{(2i+1/\lambda)(2j+1/\lambda)}}{\lambda\Gamma(2-2\mu)}\sum_{m=0}^N\sum_{n=0}^M\Big(\frac{1-\hat{\zeta}_n^{\frac{1}{\lambda}}}{1-\hat{\zeta}_n}\Big)^{1-2\mu}J_j^{0,1/\lambda-1}(2\zeta_m\hat{\zeta}_n-1)\hat{\omega}_n\\ &\qquad J_i^{0,1/\lambda-1}(2\zeta_m-1)\omega_m,\\ F_{i,j}^{k+1}=&\big(f(x,y,t_{k+1/2}),{}^+\varphi_i^{\lambda_1}(x){}^+\varphi_j^{\lambda_2}(y)\big),\end{aligned}$$

with $i=1,2,\cdots,N_1$; $j=1,2,\cdots,N_2$, $\{\xi_k\}_{k=0}^N$, $\{\zeta_m\}_{m=0}^N$, and $\{\hat{\zeta}_n\}_{n=0}^M$ are zeros of the shifted Jacobi polynomials $J_{N+1}^{2,3/\lambda-1}(2x-1)$, $J_{N+1}^{\frac{3-2\mu}{\lambda}-1,0}(2x-1)$, and $J_{M+1}^{1-2\mu,\frac{1}{\lambda}-1}(2x-1)$ respectively, $\{\rho_k\}_{k=0}^N$, $\{\omega_m\}_{m=0}^N$, and $\{\hat{\omega}_n\}_{n=0}^M$ are respectively the associated Gauss weights. The evaluation of $F_{i,j}^k$ can be either exact or approximative, depending on how smooth is f used in the actual calculation.

6 Numerical results

We present numerical results to verify the error estimates obtained in Theorem 4.1 for the proposed time stepping scheme and Müntz spectral method. Theorem 4.1 indicates that the convergence of numerical solutions would be of 2-order accuracy in time and exponential in space with respect to the Müntz polynomial degree if the exact solution or its transformation has certain regularity. In all calculation that follows, we set p, q:

$$p=\begin{cases} 1, & 0\le\mu_1<\dfrac{1}{2}, \\ -1, & \dfrac{1}{2}<\mu_1<1, \end{cases} \qquad q=\begin{cases} 1, & 0\le\mu_2<\dfrac{1}{2}, \\ -1 & \dfrac{1}{2}<\mu_2<1. \end{cases}$$

Example 6.1. We consider the problem (2.1) with fabricated exact solution:

$$u(x,y,t)=\cos(\pi t)\sin\Big(\frac{1}{2}\pi x^{\alpha}(1-x)\Big)\sin(2\pi y^{\beta}(1-y)),$$

where α,β are positive real numbers.

First we investigate the time accuracy. In Table 1, we present the errors of the computed solution as a function of the time step size τ for some fixed μ_1,μ_2 with $\nu=1$, $\alpha=1$, $\beta=2$ and $N_1=N_2=20$. As predicted by the estimate in Theorem 3.2, the proposed time stepping scheme yields a temporal approximation order close to 2.

Then we study the accuracy of the new spectral method. In the first test case, we consider a smooth exact solution corresponding to $\alpha=1$, $\beta=2$. In this case, we take $\lambda_1=\lambda_2=1$ in the discrete problem (4.1), corresponding to the standard spectral methods using classical polynomials, which is a natural choice when the solution is smooth. The spatial accuracy is shown in Fig. 1 with $\nu=1$ by fixing a small time step size $\tau=10^{-4}$. In Fig. 1(a)-(b), we plot the L^2-, H^1-, L^∞-errors in semi-log scale with respect to the polynomial degrees N_1 for $2\mu_1=0.1$, 1.1 with fixed $2\mu_2=1.1$ and $N_2=30$. The fact that all

Table 1: Errors versus time step size τ for some different μ_1,μ_2 with $\nu=1$, $\alpha=1$, $\beta=2$ and $N_1=N_2=20$ at $T=1$.

	$2\mu_1=2\mu_2=0.1$		$2\mu_1=2\mu_2=1.1$		$2\mu_1=2\mu_2=1.9$	
τ	H^1-norm	order	H^1-norm	order	H^1-norm	order
$\frac{1}{10}$	9.48132563e-03	–	9.43110000e-03	–	7.42386642e-03	–
$\frac{1}{20}$	2.33148483e-03	2.02383	2.32333005e-03	2.02123	1.84526227e-03	2.00834
$\frac{1}{40}$	5.81329136e-04	2.00382	5.79399208e-04	2.00356	4.60406184e-04	2.00284
$\frac{1}{80}$	1.45252801e-04	2.00078	1.44773406e-04	2.00076	1.14836008e-04	2.00333
$\frac{1}{160}$	3.63085749e-05	2.00018	3.61888406e-05	2.00017	2.85517377e-05	2.00792
$\frac{1}{320}$	9.07687271e-06	2.00004	9.04694120e-06	2.00004	7.28243543e-06	1.97108

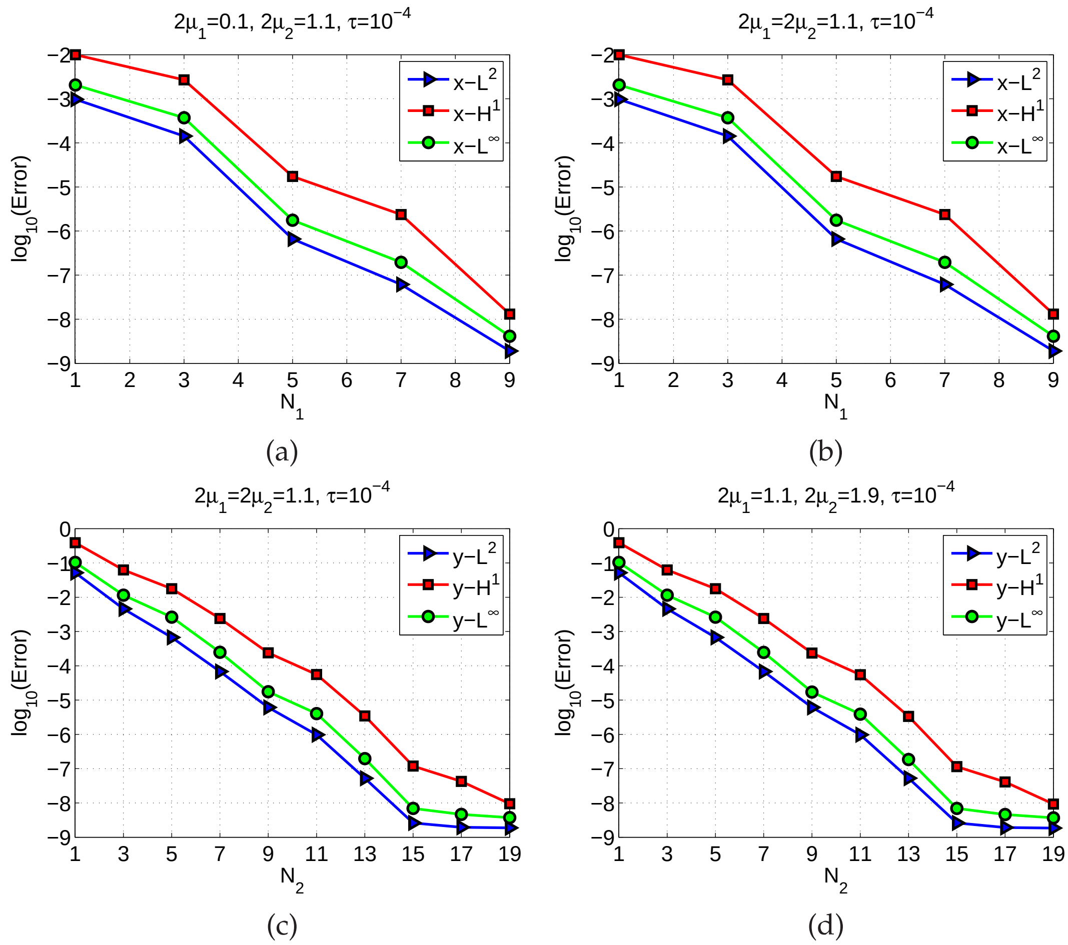

Figure 1: Error decays of the numerical solutions with respect to the polynomial degrees for the smooth exact solution with $\nu=1$, $\alpha=1$, $\beta=2$, $\lambda_1=\lambda_2=1$. (a) $2\mu_1=0.1$, $2\mu_2=1.1$, $N_2=30$; (b) $2\mu_1=2\mu_2=1.1$, $N_2=30$; (c) $2\mu_1=2\mu_2=1.1$, $N_1=20$; (d) $2\mu_1=1.1$, $2\mu_2=1.9$, $N_1=20$.

error curves are straight lines in these semi-log plots indicates that the numerical solutions are exponentially convergent with respect to N_1. Here "x-L^2", "x-H^1" and "x-L^∞" in the figures mean L^2-, H^1- and L^∞-errors in the x-direction. Similar notations apply for the errors in the y-direction in Fig. 1(c)-(d), showing the error behavior as a function of the polynomial degree N_2 for $2\mu_2=1.1$, 1.9 with fixed $2\mu_1=1.1$ and $N_1=20$. As expected, the numerical solutions are also exponentially convergent with respect to the polynomial degree in the y-direction.

The next test case is for non-smooth solutions, which is used to verify the efficiency and flexibility of the proposed method using non traditional polynomials. Fig. 2 shows the error decay of the numerical solutions versus the polynomial degrees in different directions for a number of values of μ_1, μ_2, λ_1, λ_2, etc. Once again, all error curves

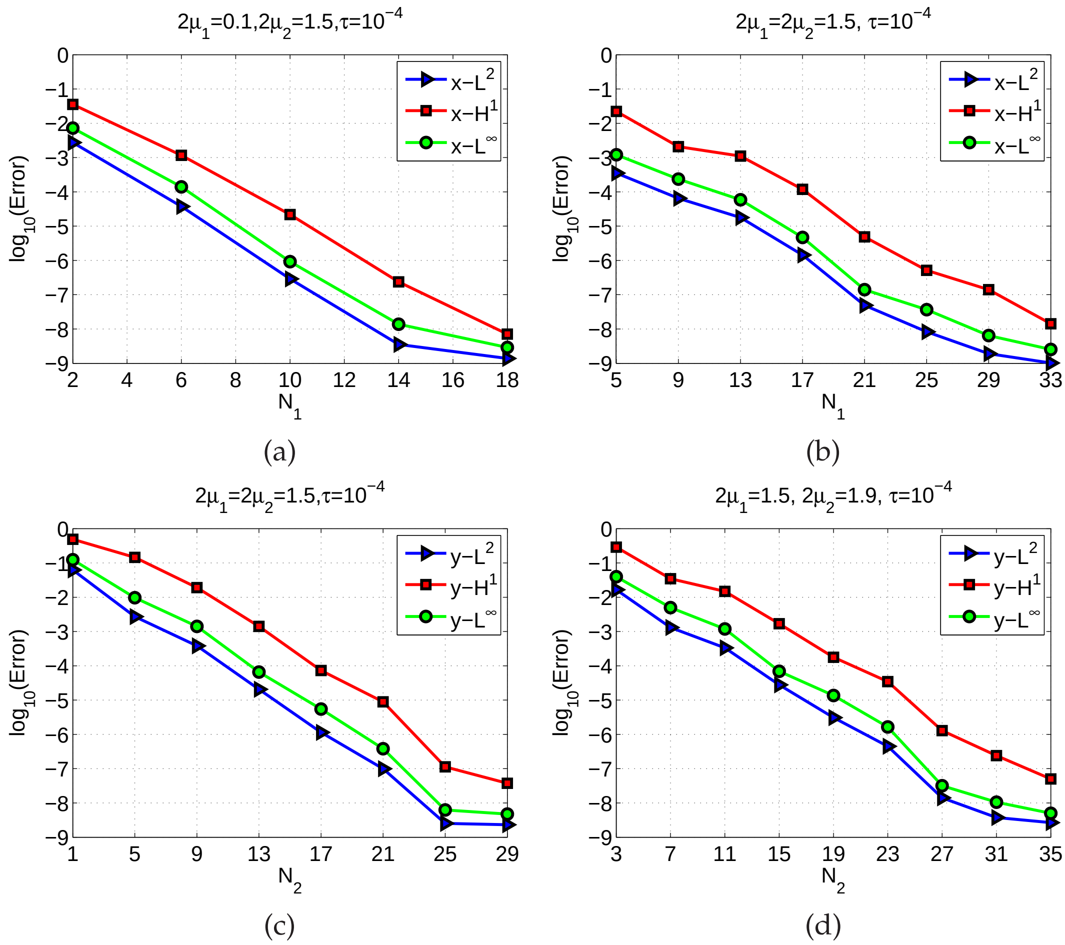

Figure 2: Error decays of the numerical solutions versus the polynomial degrees for the non-smooth exact solution with $\nu=1$. (a) $2\mu_1=0.1$, $2\mu_2=1.1$, $\alpha=1.5$, $\beta=2$, $\lambda_1=\frac{1}{2}$, $\lambda_2=1$, $N_2=40$; (b) $2\mu_1=2\mu_2=1.5$, $\alpha=1.1$, $\beta=2$, $\lambda_1=\frac{1}{10}$, $\lambda_2=1$, $N_2=40$; (c) $2\mu_1=2\mu_2=1.5$, $\alpha=1$, $\beta=\frac{5}{3}$, $\lambda_1=1$, $\lambda_2=\frac{1}{3}$, $N_1=40$; (d) $2\mu_1=1.1$, $2\mu_2=1.9$, $\alpha=1$, $\beta=1.2$, $\lambda_1=1$, $\lambda_2=\frac{1}{5}$, $N_1=40$.

are straight line with respect to N_1 or N_2 in these semi-log plots. This clearly indicates that the Müntz spectral methods is exponential convergent as long as suitable λ_1, λ_2 are used even if the exact solution is not smooth. This observation is in perfect agreement with the theoretical prediction given in the main theorem, stating that the convergence of numerical solution is exponential if suitable variable changes make the solution smooth. It is notable that some similar results have been observed (not shown in the paper) for the case $\nu=0$.

In the second example, we consider problem (2.1) with an arbitrary smooth force function f. In this case, the exact solution of (2.1) is unknown, and the numerical solution computed with a large enough N will be served as the "exact" solution. In what follows, we will focus our attentions on the spatial error behaviors by addressing a time independent problem.

Example 6.2. Consider the steady state problem (2.1):

$$\begin{cases} u-\nu\Delta u+p\,{}_0D_x^{2\mu_1}u+q\,{}_0D_y^{2\mu_2}u=f(x,y), & (x,y)\in\Omega, \\ u|_{\partial\Omega}=0, \end{cases} \tag{6.1}$$

with some smooth forcing function $f(x,y)$.

We first test the method for $p=q=0$, $f\equiv 1$. The "exact" solution is the numerical solution calculated with $N_1=120$, $N_2=120$. The pointwise error of the numerical solution computed by the traditional spectral method ($\lambda_1=\lambda_2=1$) with $N_1=N_2=30$ is plotted in Fig. 3(a), while the decay rate of the error in different norms is shown in Fig. 3(b). It is seen that larger error appears around the corners of the domain, which is due to the well known corner singularity of the elliptic problem. The straightline of the error curves in log-log representation in the right figure indicates that only algebraic accuracy is achieved when the traditional polynomials are used.

Now we consider $f(x,y)=\sin(\pi x)sin(\pi y)$. The good feature of this f, i.e., $f(0,0)=f(0,1)=f(1,0)=f(1,1)=0$, will weaken the corner singularity associated to the Laplace operator [21]. This is demonstrated in Fig. 4(a), showing that the exponential convergence is obtained by the traditional spectral method in the case the fractional terms are absent in the equation (i.e., $\nu=1$, $p=q=0$). This means that the main singularity of (6.1) with $\nu\neq 1$, $p\neq 0$, $q\neq 0$ will be located at the corner $(0,0)$ due to presence of the fractional operators. This singularity has been known in some 1D fractional differential equations; see, e.g., [13, 32]. So, although to our knowledge there is no available

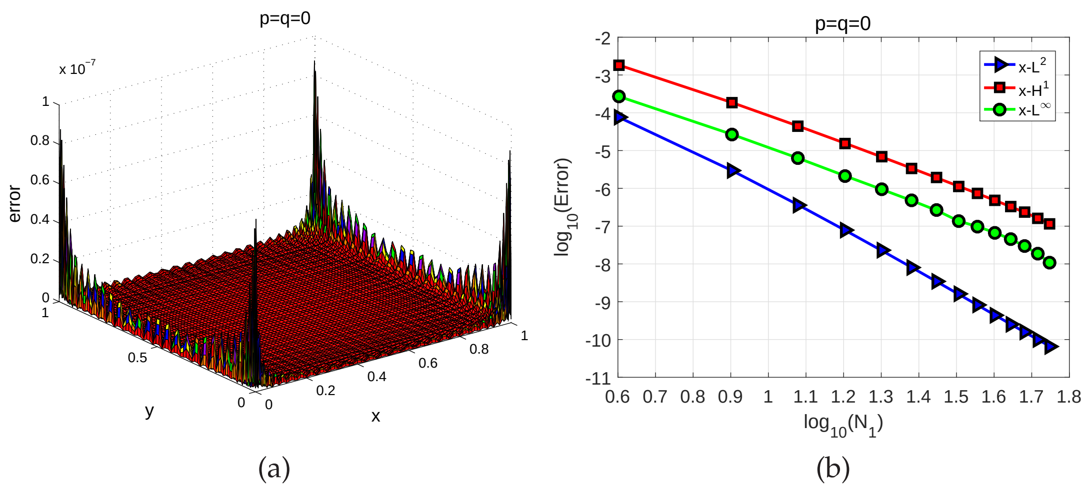

Figure 3: Error behavior of the traditional spectral method, i.e., $\lambda_1=\lambda_2=1$ for $p=q=0$, $f\equiv 1$. (a) pointwise error with $N_1=N_2=30$; (b) error decays versus N_1 with fixed $N_2=80$.

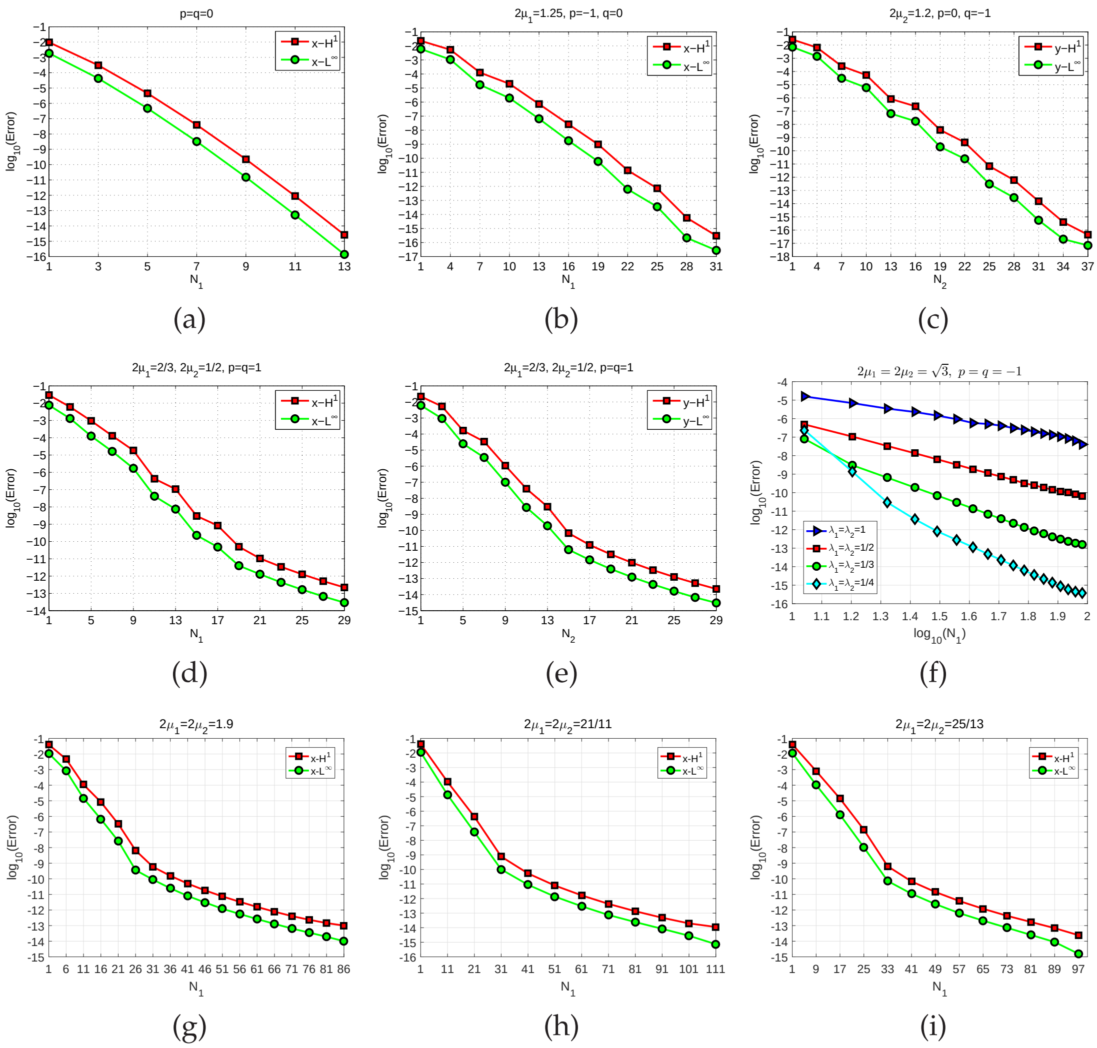

Figure 4: Error behavior with respect to the polynomial degrees with $f(x,y)=\sin(\pi x)\sin(\pi y)$. (a) $\nu=1$, $p=q=0$, $\lambda_1=\lambda_2=1$ with $N_2=40$; (b) $\nu=1$, $p=-1$, $q=0$, $2\mu_1=1.25$, $\lambda_1=\frac{1}{4}$, $\lambda_2=1$ with $N_2=40$; (c) $\nu=1$, $p=0$, $q=-1$, $2\mu_2=1.2$, $\lambda_1=1$, $\lambda_2=\frac{1}{5}$ with $N_1=40$; (d) $\nu=1$, $2\mu_1=\frac{2}{3}$, $2\mu_2=\frac{1}{2}$, $\lambda_1=\frac{1}{3}$, $\lambda_2=\frac{1}{2}$ with $N_2=40$; (e) $\nu=1$, $2\mu_1=\frac{2}{3}$, $2\mu_2=\frac{1}{2}$, $\lambda_1=\frac{1}{3}$, $\lambda_2=\frac{1}{2}$ with $N_1=40$; (f) $\nu=1$, $2\mu_1=2\mu_2=\sqrt{3}$ with different λ_1, λ_2 and fixed $N_2=100$ in L^∞-norm; (g) $\nu=0$, $2\mu_1=2\mu_2=1.9$, $\lambda_1=\lambda_2=\frac{1}{10}$ with $N_2=100$; (h) $\nu=0$, $2\mu_1=2\mu_2=\frac{21}{11}$, $\lambda_1=\lambda_2=\frac{1}{11}$ with $N_2=120$; (i) $\nu=0$, $2\mu_1=2\mu_2=\frac{25}{13}$, $\lambda_1=\lambda_2=\frac{1}{13}$ with $N_2=100$.

theoretical analysis, it is reasonable to conjecture that the two-dimensional FCDE with $f(x,y)=\sin(\pi x)sin(\pi y)$ has similar singularity around the corner $(0,0)$. This singularity can be represented by the Mittag-Leffle function as discussed in [13, Lemma 2.6]. In the following tests, we adopt the strategy given in [13, Remark 5.5] in choosing λ_1 and λ_2. The computed results for different combinations of the parameters $p,q,\mu_1,\mu_2,\lambda_1$, λ_2, etc, are presented in Fig. 4. Fig. 4(b)-(e) and (g)-(i) show the error curves for the case of rational μ_1 and μ_2. Once again, we observe the spectral accuracy for the new spectral method with suitably chosen λ_1 and λ_2 in Fig. 4(b) and (c). However, the errors shown in Fig. 4(d)-(e) and (g)-(i) decay, although of high order, only with algebraic convergence rate. This is probably due to the fact that the solution has singularity structure like $(x^2+y^2)^\rho$ for some real number ρ, which could not be made smooth through separate variable changes $x\to x^{1/\lambda_1}$ and $y\to y^{1/\lambda_2}$. Even so, one can still try to use moderate λ to accelerate the convergence rate as discussed in [13, Remark 5.5]. In fact, the numerical results presented in Fig. 4(f) in the current paper confirm this point.

7 Concluding remarks

We have developed and analyzed an efficient numerical method for the two-dimensional space-fractional convection-diffusion equation. The proposed method made use of the Crank-Nicolson scheme for the temporal discretization and a new spectral method using the Müntz Jacobi polynomials for the spatial discretization. The main contribution of the paper is: the Müntz spectral method for the 1D time-fractional diffusion equation introduced in [13,15] was generalized to the two-dimensional space-fractional convection-diffusion equation with a new class of Müntz Jacobi polynomials. Compared to the 1D time-fractional diffusion equation, some new operators including H^1-projectors were defined, and more technique were needed in order to derive satisfying error estimates. A detailed stability and convergence analysis was carried out to show that the time stepping scheme is of unconditionally stable and second order convergent. The obtained error estimates for the spatial approximation have shown that the new method is capable to achieve spectral convergence for the solution $u(x,y,t)$ with $\partial_{xy}u(\cdot^{1/\lambda_1},\cdot^{1/\lambda_2},t)$ being smooth enough. A series of numerical experiments were performed to demonstrate the efficiency of the method.

It is worthy to mention that the present method is limited to handle one corner singularity, which has already been found useful in a large class of 2D fractional PDEs. In the future we plan to integrate the proposed method into the spectral element framework, so that it can treat all corner singularity.

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