Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 6, pp. 1398-1414

DOI: 10.4208/aamm.OA-2018-0267
December 2019

Some New Generating Functions for the Modified Laguerre Polynomials

Nejla Özmen*

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Received 28 December 2018; Accepted (in revised version) 6 May 2019

Abstract. In this paper, we obtain some new results on bilateral generating functions of the modified Laguerre polynomials. We also get generating function relations between the modified Laguerre polynomials and the generalized Lauricella functions. Some special cases and important applications are also discussed.

AMS subject classifications: 33C45, 33C65

Key words: Modified Laguerre polynomials, generating function, multilinear and multilateral generating function, recurrence relations, generalized Lauricella function.

1 Introduction

In solving many problems of theoretical and mathematical physics, we often need various special functions. So, it is important to analyze a special function and to determine its characteristic properties such as its generating function relations, integral representations, explicit formula, recurrence relations, and so on. In the present paper we mainly study on the modified versions of the classical Laguerre polynomials that are famous ones in the theory of special functions. More precisely, we obtain some new results regarding bilateral generating functions of the modified Laguerre polynomials.

In the literature there are various methods of obtaining generating functions, such as group-theoretic method and analytic method (see, for instance, [3,4,7,18,29,31]).

Throughout the paper we need the following definitions and notations.

The modified Laguerre polynomials, denoted by $f_n^{(\beta)}(x)$, are specified by the series (see [12])

$$f_n^{(\beta)}(x)=\frac{(\beta)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(1-n-\beta)_k}\frac{x^k}{k!}, \tag{1.1}$$

*Corresponding author.
Emails: nejlaozmen06@gmail.com, nejlaozmen@duzce.edu.tr (N. Özmen)

where, as usual, $(\lambda)_\nu$ denotes the Pochhammer symbol given by (for $\lambda, \nu \in \mathbb{C}$ and in terms of Gamma function)

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu=0,\ \lambda\in\mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (\nu=n\in\mathbb{N},\ \lambda\in\mathbb{C}), \end{cases}$$

with the convention $(0)_0 := 1$.

These polynomials have the following generating function relations (see [18]):

$$\sum_{n=0}^{\infty} f_n^{(\beta)}(x) t^n = (1-t)^{-\beta} \exp(xt), \qquad |t|<1, \tag{1.2a}$$

$$\sum_{n=0}^{\infty} \binom{n+m}{m} f_{n+m}^{(\beta)}(x) t^n = (1-t)^{-\beta-m} \exp(xt) f_m^{(\beta)}(x(1-t)), \qquad |t|<1. \tag{1.2b}$$

Another modified version of the Laguerre polynomials of degree n, denoted by $L_{\alpha,\beta,m,n}(x)$, was defined by Goyal [13] in the form

$$L_{\alpha,\beta,m,n}(x) = \frac{\beta^n (m)_n}{n!}\, {}_1F_1\Big(-n;m;\frac{\alpha x}{\beta}\Big), \qquad (m\neq 0,-1,-2,\cdots,\quad \beta\neq 0), \tag{1.3}$$

where ${}_1F_1$ is the confluent hypergeometric function (see [10]). We should note that the polynomials $L_{\alpha,\beta,m,n}(x)$ are found in an increasing number of mathematical works since it is more easier to handle and also more practical in numerical computations (see, for instance, [17, 19, 26–28]). For the polynomials in (1.3), the following generating function was obtain in [24]:

$$\sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} L_{\alpha,\beta,m,n+k}(x) t^n = (1-\beta t)^{-m-k} \exp\left(\frac{\alpha x t}{\beta t-1}\right) L_{\alpha,\beta,m,k}\left(\frac{x}{1-\beta t}\right), \tag{1.4}$$

where $|\beta t|<1$. In recent years, many researchers have studied multilinear and multilateral generating functions for different type of polynomials, such as Altın et al. [1], Chan et al. [2], Chen et al. [5,6], Dattoil et al. [8], Erkus et al. [11] and Liu [14], Qureshi et al. [25]. Similarly, in [15] Liu et al. introduced bilateral generating functions for the Chan-Chyan-Srivastava polynomials and the generalized Lauricella functions, and in [16] bilateral generating functions for the Erkus-Srivastava polynomials and generalized Lauricella functions were derived. Recently, we have obtained generating functions for the generalized Lauricella polynomials and the generalized Cesáro functions (see [23]). It is also possible to find different generating functions by means of a similar method used in [20] and [22].

With the above information, our strategy in this paper is as follows: in Section 2, we get an integral representation of the polynomials $f_n^{(\beta)}(x)$ and provide a new generating function relation for the polynomials $L_{\alpha,\beta,m,n}(x)$. In Section 3, we derive several families of bilinear and bilateral generating functions for the polynomials $L_{\alpha,\beta,m,n}(x)$. In Section 4, we discuss some special cases. In Section 5, we obtain various families of bilateral generating functions for the modified Laguerre polynomials and the generalized Lauricella functions defined in [30]. The last section is devoted to the concluding remarks.