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Interior Penalty Discontinuous Galerkin Based Isogeometric Analysis for Allen-Cahn Equations on Surfaces

Futao Zhang*, Yan Xu, Falai Chen and Ruihan Guo

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.

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Abstract. We propose a method that combines Isogeometric Analysis (IGA) with the interior penalty discontinuous Galerkin (IPDG) method for solving the Allen-Cahn equation, arising from phase transition in materials science, on three-dimensional (3D) surfaces consisting of multiple patches. DG ideology is adopted at patch level, i.e., we employ the standard IGA within each patch, and employ the IPDG method across the patch interfaces. IGA is very suitable for solving Partial Differential Equations (PDEs) on (3D) surfaces and the IPDG method is used to glue the multiple patches together to get the right solution. Our method takes advantage of both IGA and the IPDG method, which allows us to design a superior semi-discrete (in time) IPDG scheme. First and most importantly, the time-consuming mesh generation process in traditional Finite Element Analysis (FEA) is no longer necessary and refinements, including h-refinement and p-refinement which both maintain the original geometry, can be easily performed at any level. Moreover, the flexibility of the IPDG method makes our method very easy to handle cases with non-conforming patches and different degrees across the patch interfaces. Additionally, the geometrical error is eliminated (for all conic sections) or significantly reduced at the beginning due to the geometric flexibility of IGA basis functions, especially the use of multiple patches. Finally, this method can be easily formulated and implemented. We present our semi-discrete IPDG scheme after generally describe the problem, and then briefly introduce the time marching method employed in this paper. Theoretical analysis is carried out to show that our method satisfies a discrete energy law, and achieves the optimal convergence rate with respect to the L^2 norm. Furthermore, we propose an elliptic projection operator on (3D) surfaces and prove an approximation error estimate which are vital for us to obtain the error estimate in the L^2 norm. Numerical tests are given to validate the theory and gauge the good performance of our method.

AMS subject classifications: 65M60, 65D17, 68U07, 35K93

*Corresponding author. *Email addresses:* zhft00@mail.ustc.edu.cn (F. Zhang), yxu@ustc.edu.cn (Y. Xu), chenfl@ustc.edu.cn (F. Chen), guoguo88@mail.ustc.edu.cn (R. Guo)

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1 Introduction

This paper is concerned with the IPDG based IGA (see the discussion below) for the Allen-Cahn equation on (3D) surfaces:

$$\begin{cases} u_t-\Delta_s u+\frac{1}{\varepsilon^2}f(u)=0, & (\boldsymbol{x},t)\in\mathfrak{D}\times[0,T], \\ \nabla_s u\cdot\boldsymbol{n}=0, & (\boldsymbol{x},t)\in\partial\mathfrak{D}\times[0,T], \\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}), & \boldsymbol{x}\in\mathfrak{D}, \end{cases} \tag{1.1}$$

where Δ_s and ∇_s denote the Laplace-Beltrami operator and the tangent gradient operator on surface, respectively. $\mathfrak{D}\subset\mathbb{R}^3$ is a bounded surface domain with piecewise smooth boundary $\partial\mathfrak{D}$. $\boldsymbol{n}$, the unit outward normal of $\mathfrak{D}$ along $\partial\mathfrak{D}$, is defined from a cross product of the tangent vector of $\partial\mathfrak{D}$ and the unit normal vector of $\mathfrak{D}$ on $\partial\mathfrak{D}$. T is a fixed constant and $f(u)=\mathcal{F}'(u)$ for some double well potential density function $\mathcal{F}$, which takes its global minimum value 0 at $u=\pm1$. In this paper, we focus on the following widely used quartic density function:

$$\mathcal{F}(u)=\frac{1}{4}(u^2-1)^2. \tag{1.2}$$

Note that (1.1) differs from the original Allen-Cahn equation (see [2]) in the scaling of the time and domain. Actually, the time t here, called the *fast time*, represents t/ε^2 in the original formulation. In addition, (1.1) is defined on a (3D) surface domain $\mathfrak{D}$.

The original Allen-Cahn equation was introduced by Allen and Cahn [2] to describe the motion of antiphase boundaries in crystalline solids. It was proposed as a simple (nonconservative) model for the process of phase separation of a binary alloy at a fixed temperature (cf. [31] and the reference therein), where the function u represents the concentration of one of the two metallic components of the alloy and the parameter ε is an *interaction length*, which is small compared to the characteristic dimensions on the laboratory scale. Recently, it has been applied to a wide range of problems such as the motion by mean curvature flows [24] and crystal growth [35]. In particular, it has become a basic model equation for the diffuse interface approach developed to study phase transitions and interfacial dynamics in materials science [11]. For more physical background, derivation, and discussion of the Allen-Cahn equation, we refer the reader to [2,12,19,21,27].

Many numerical methods and corresponding theoretical analysis have been developed to solve the Allen-Cahn equation for two-dimensional (2D) plane case and (3D) volume case, see [5,13,14,25,26,28,38,40]. However, applications in practice always need to solve the Allen-Cahn equation on (3D) surfaces and no such work is carried out as far as we know. Of course, the traditional FEA could be used to handle this problem, we refer readers to the survey paper [20] for details about different finite element methods