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# Multiplicative Jordan Decomposition in Integral Group Ring of Group $K_8 \times C_5$

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**Abstract:** In this article, we present the multiplicative Jordan decomposition in integral group ring of group $K_8 \times C_5$, where $K_8$ is the quaternion group of order 8. Thus, we give a positive answer to the question raised by Hales A W, Passi I B S and Wilson L E in the paper "The multiplicative Jordan decomposition in group rings II. *J. Algebra*, 2007, **316**: 109–132".

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## 1 Introduction

Let $G$ be a finite group and $\mathbf{Q}$ the field of rational numbers. Then every element $\alpha$ in the group algebra $\mathbf{Q}G$ has a unique additive Jordan decomposition $\alpha = \alpha_s + \alpha_n$ with $\alpha_s, \alpha_n \in \mathbf{Q}G$, $\alpha_s$ is semisimple, $\alpha_n$ is nilpotent and $\alpha_s\alpha_n = \alpha_n\alpha_s$. Recall that an element $\alpha \in FG$ is said to be semisimple if the minimal polynomial $m(X)$ of $\alpha$ over $F$ does not have repeated roots in the algebraic closure $\overline{F}$ of $F$ with $F$ a field of characteristic 0. Furthermore, if $\alpha$ is a unit in $\mathbf{Q}G$, then so is $\alpha_s$, and $\alpha$ has a unique multiplicative Jordan decomposition $\alpha = \alpha_s\alpha_u$ with $\alpha_u = 1+\alpha_s^{-1}\alpha_n$ unipotent and $\alpha_s\alpha_u = \alpha_u\alpha_s$. But, when $\alpha \in \mathbf{Z}G$, the integral group ring over $G$, the semisimple component $\alpha_s$ does not always lie in $\mathbf{Z}G$. The integral group ring $\mathbf{Z}G$ is said to have the additive Jordan decomposition (or AJD for short) property if $\alpha_s \in \mathbf{Z}G$ (and hence $\alpha_n \in \mathbf{Z}G$) for every $\alpha \in \mathbf{Z}G$, and to have the multiplicative Jordan decomposition (or MJD for short) property if $\alpha_s$ and $\alpha_u \in \mathbf{Z}G$ for every unit $\alpha \in \mathbf{Z}G$. If $\mathbf{Z}G$ has the AJD property, then in fact it also has the MJD property. Therefore, to consider

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the groups $G$ such that $\mathbf{Z}G$ has the MJD property, we need to consider the case that if $\mathbf{Z}G$ has the AJD property.

The finite groups $G$ whose integral group ring $\mathbf{Z}G$ has the AJD property are completely classified in [1] and [2]. If $G$ is abelian or a Hamiltonian 2-group, then every element of $\mathbf{Q}G$ is semisimple and consequently $\mathbf{Z}G$ trivially has the MJD property. In [3], the necessary conditions for a finite group $G$ whose integral group ring $\mathbf{Z}G$ has the MJD property are given as follows:

**Theorem 1.1** ([3], Theorem 29) *Let $G$ be a finite group such that $\mathbf{Z}G$ has MJD. Then one of the following holds*:

(1) *$G$ is either abelian or of the form $K_8 \times E \times H$, where $E$ is an elementary abelian 2-group and $H$ is abelian of odd order so that 2 has odd multiplicative order mod $|H|$. (Such $G$ have AJD and hence MJD for trivial reasons, since $\mathbf{Q}G$ contains no nilpotent.)*

(2) *$G$ has order $2^a 3^b$ for some nonnegative integers $a$, $b$.*

(3) *$G = K_8 \times C_p$ for some prime $p \geq 5$ so that 2 has even multiplicative order mod $p$.*

(4) *$G$ is the split extension of $C_p$ $(p \geq 5)$ by a cyclic group $\langle g \rangle$ of order $2^k$ or $3^k$ for some $k \geq 1$, and $g^2$ or $g^3$ acts trivially on $C_p$.*

To completely classify the finite groups $G$ such that $\mathbf{Z}G$ has the MJD property, we need only to investigate the four cases listed in Theorem 1.1. It has been shown that the integral group rings of abelian groups have the AJD property in [2], and so the MJD property. For the finite non-abelian 2-groups whose integral group rings possess the MJD property, there are two groups of order 8 (see [4]), five groups of order 16 (see [5]), four groups of order 32 and only the Hamiltonian groups of larger order (see [3] for details). Liu and Passman[6] showed that for the finite non-abelian 3-groups whose integral group rings have the MJD property, there are two groups of order 27 and at most three other groups (all of order 81) for larger order. Liu and Passman[7] also proved that there are precisely three non-abelian 2,3-groups of order divisible by 6, with $\mathbf{Z}G$ satisfying MJD.

Since $\mathbf{Q}(K_8 \times C_p)$ has no nilpotent elements for $p$ a prime such that 2 has odd multiplicative order mod $p$, the integral group ring $\mathbf{Z}(K_8 \times C_p)$ trivially has the MJD property. When $p$ is some odd prime such that 2 has even multiplicative order mod $p$, Hales *et al.*[3] claimed that "We do not know if these groups ever have MJD for their integral group rings. The first example to investigate would be $\mathbf{Z}(K_8 \times C_5)$."

In this article, we present the multiplicative Jordan decomposition in integral group ring of group $K_8 \times C_5$, where $K_8$ is the quaternion group of order 8. Thus, we give a positive answer to the question raised by Hales *et al.* in [3].

# 2 The AJD for the Units of $\mathbf{Z}(K_8 \times C_p)$ in $\mathbf{Q}(K_8 \times C_p)$

**Lemma 2.1**[1] *If $\alpha$ in $\mathbf{Q}G$ is central and $\beta$ in $\mathbf{Q}G$ is semisimple, then $\alpha + \beta$ is semisimple.*

Let $C_p = \langle t \rangle$ be a cyclic group of order $p$ and $\zeta$ a primitive $p$th root of unity. Let $U_1(\mathbf{Z}C_p)$ and $U_1(\mathbf{Z}[\zeta])$ denote the sets of 1-units for $\mathbf{Z}C_p$ and $\mathbf{Z}[\zeta]$, separately. We consider the map