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# A ROBUST INTERIOR POINT METHOD FOR COMPUTING THE ANALYTIC CENTER OF AN ILL-CONDITIONED POLYTOPE WITH ERRORS*

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**Abstract**

In this paper we propose an efficient and robust method for computing the analytic center of the polyhedral set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$, where the matrix $A \in R^{m\times n}$ is ill-conditioned, and there are errors in $A$ and $b$. Besides overcoming the difficulties caused by ill-conditioning of the matrix $A$ and errors in $A$ and $b$, our method can also detect the infeasibility and the unboundedness of the polyhedral set $P$ automatically during the computation. Detailed mathematical analyses for our method are presented and the worst case complexity of the algorithm is also given. Finally some numerical results are presented to show the robustness and effectiveness of the new method.

*Mathematics subject classification:* 65K05, 90C51.
*Key words:* Analytic center, Ill-conditioning, Unboundedness, Primal-dual interior point algorithm, Convergence, Polynomial complexity.

## 1. Introduction

The computation of the analytic center of a polytope has been discussed in many papers and books as the development of interior point methods (IPMs) for linear programming (LP), see, e. g., [1,9,30] and the references therein. In this paper, we will consider the following problem:

$$\begin{aligned} \min \quad & -\sum_{j=1}^{n} \ln x_j \\ \text{s.t.} \quad & Ax = b, \\ & x_j > 0, \quad j = 1, \ldots, n, \end{aligned} \tag{1.1}$$

where the matrix $A \in R^{m\times n}$ is ill-conditioned, $b \in R^m$, and there are errors in $A$ and $b$ which may lead to an infeasible problem. Problem (1.1) comes from the study of determining the neutron energy spectrum from multiple activation foils in nuclear physics which is based on the

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maximum entropy principle of thermodynamics and Boltzmann's entropy formula, see [6, 35] for more details.

In fact, problem (1.1) is equivalent to the computation of the analytic center of the polyhedral set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ when $P$ is solid (i.e., the set has at least one positive feasible point) and bounded. Theory of IPMs for LP has established abundant results on the computation of analytic center (see, e.g., [5, 24, 28, 30] and the references therein). If we solve the problem (1.1) directly by some kind of interior point method, a linear system $AD^2A^Ty = h$ or some equivalent linear systems will be formed with ill-conditioned matrices $A$ and $D$. Although M. H. Wright [27] and S. J. Wright [29] have shown that the ill-conditioning of matrix $D$ does not noticeably impair the accuracy of the computed primal-dual steps in certain cases, the ill-conditioning of matrix $A$ can cause serious numerical difficulties during computation, let alone the errors in $A$ and $b$. We have used some optimization tools in MATLAB which are based on LIPSOL [34] and IPOPT [25] to solve some instances of problem (1.1), but their numerical performances are unsatisfactory (which will be shown by numerical experiments in Section 4). Despite the fact that some precondition techniques (see, e.g., [19, 20] and references therein) can effectively reduce the ill-conditioning of matrix $A$, the errors in $A$ and $b$ will still prevent problem (1.1) from being solved effectively.

Based on the primal-dual infeasible-interior-point (IIP) method, we will propose an efficient and robust method for problem (1.1) which can overcome the difficulties caused by ill-conditioning of the matrix $A$ and errors in $A$ and $b$. The paper is organized as follows. In Section 2 we reformulate problem (1.1) into a problem which is equivalent to finding the analytic center of the optimal solution set of a usually well-conditioned linear programming (LP) problem. The reformulation not only can reduce the impact of errors in $A$ and $b$ but also can improve the condition number and detect the infeasibility of the set $P$. Then in Section 3 we will consider a hybrid primal-dual IIP algorithm for computing the analytic center of the optimal solution set of an LP problem. Moreover, we will study a method in detail for detecting unboundedness or infeasibility of the set $P$ which may cause computational difficulty for general interior point algorithms. The convergent properties and the worst case complexity of the algorithm are also analysed. Finally some numerical results are presented to show the effectiveness and robustness of our algorithm in Section 4, and some conclusions and remarks are given in Section 5.

## 2. Reformualtion

If the polyhedral set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ is unbounded or not solid, problem (1.1) will have no optimal solution. Even when the set $P$ is solid and bounded, the ill-conditioning of matrix $A$ will cause serious numerical difficulties in computation, and errors in $A$ and $b$ will prevent us from using precondition techniques directly. Hence some reformulation of problem (1.1) is needed.

Various reformulations for ill-posed problems have been proposed in literature, such as Tikhonov regularization method and trust region method (see, e.g., [26]). However, these methods can not be used directly to problem (1.1) because of the errors. Hence we consider the following first phase problem for LP:

$$\begin{aligned} \min \quad & e^Ty \equiv \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & Ax + y = b, \\ & x \geq 0,\ y \geq 0. \end{aligned} \tag{2.1}$$