Journal of Computational Mathematics
Vol.37, No.6, 2019, 739–757.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1803-m2018-0278

A UNIFIED ALGORITHMIC FRAMEWORK OF SYMMETRIC GAUSS-SEIDEL DECOMPOSITION BASED PROXIMAL ADMMS FOR CONVEX COMPOSITE PROGRAMMING*

Liang Chen
College of Mathematics and Econometrics, Hunan University, Changsha 410082, China;
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China
Email: chl@hnu.edu.cn
Defeng Sun
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China
Email: defeng.sun@polyu.edu.hk
Kim-Chuan Toh
Department of Mathematics, and Institute of Operations Research and Analytics,
National University of Singapore, 10 Lower Kent Ridge Road, Singapore
Email: mattohkc@nus.edu.sg
Ning Zhang[1)]
School of Computer Science and Technology, Dongguan University of Technology, Dongguan, China;
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China
Email: ningzhang_2008@yeah.net

Abstract

This paper aims to present a fairly accessible generalization of several symmetric Gauss-Seidel decomposition based multi-block proximal alternating direction methods of multipliers (ADMMs) for convex composite optimization problems. The proposed method unifies and refines many constructive techniques that were separately developed for the computational efficiency of multi-block ADMM-type algorithms. Specifically, the majorized augmented Lagrangian functions, the indefinite proximal terms, the inexact symmetric Gauss-Seidel decomposition theorem, the tolerance criteria of approximately solving the subproblems, and the large dual step-lengths, are all incorporated in one algorithmic framework, which we named as sGS-imiPADMM. From the popularity of convergent variants of multi-block ADMMs in recent years, especially for high-dimensional multi-block convex composite conic programming problems, the unification presented in this paper, as well as the corresponding convergence results, may have the great potential of facilitating the implementation of many multi-block ADMMs in various problem settings.

Mathematics subject classification: 90C25, 90C22, 90C06, 65K05.
Key words: Convex optimization, Multi-block, Alternating direction method of multipliers, Symmetric Gauss-Seidel decomposition, Majorization.

1. Introduction

In this paper, we consider the following multi-block convex composite programming:

$$\min_{x\in\mathcal{X},\, y\in\mathcal{Y}} \Big\{ p_1(x_1) + f(x_1,\ldots,x_m) + q_1(y_1) + g(y_1,\ldots,y_n) \,|\, \mathcal{A}^*x + \mathcal{B}^*y = c \Big\}, \qquad (1.1)$$

* Received December 18, 2018 / Accepted March 28, 2019 /
Published online July 31, 2019 /
[1)] Corresponding author.

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are three finite dimensional real Hilbert spaces, each endowed with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$, and

- $\mathcal{X}$ can be decomposed as the Cartesian product of $\mathcal{X}_1,\dots,\mathcal{X}_m$, which are finite dimensional real Hilbert spaces endowed with the inner product $\langle\cdot,\cdot\rangle$ inherited from $\mathcal{X}$ and its induced norm $\|\cdot\|$. Similarly, $\mathcal{Y}=\mathcal{Y}_1\times\cdots\times\mathcal{Y}_n$. Based on such decompositions, one can write $x\in\mathcal{X}$ as $x=(x_1,\dots,x_m)$ with $x_i\in\mathcal{X}_i,\ i=1,\dots,m$, and, similarly, $y=(y_1,\dots,y_n)$;
- $p_1:\mathcal{X}_1\to(-\infty,\infty]$ and $q_1:\mathcal{Y}_1\to(-\infty,\infty]$ are two closed proper convex functions;
- $f:\mathcal{X}\to(-\infty,\infty)$ and $g:\mathcal{Y}\to(-\infty,\infty)$ are continuously differentiable convex functions with Lipschitz continuous gradients;
- $\mathcal{A}^*$ and $\mathcal{B}^*$ are the adjoints of the two given linear mappings $\mathcal{A}:\mathcal{Z}\to\mathcal{X}$ and $\mathcal{B}:\mathcal{Z}\to\mathcal{Y}$, respectively; $c\in\mathcal{Z}$ is a given vector;
- without loss of generality, we define the two functions $p:\mathcal{X}\to(-\infty,\infty]$ and $q:\mathcal{Y}\to(-\infty,\infty]$ by $p(x):=p_1(x_1),\ \forall x\in\mathcal{X}$ and $q(y):=q_1(y_1),\ \forall y\in\mathcal{Y}$ for convenience.

At the first glance, one may view problem (1.1) as a 2-block separable convex optimization problem with coupled linear equality constraints. Consequently, the classic alternating direction method of multipliers (ADMM) [12, 13] and its contemporary variants such as [8, 10] can be used for solving problem (1.1). For the classic 2-block ADMM, one may refer to [9, 14] for a history of the algorithm and to the recent note [3] for a thorough study on its convergence properties.

In high-dimensional settings, it is usually not computationally economical to directly apply the 2-block ADMM and its variants to solve problem (1.1), as in this case solving the subproblems at each ADMM iteration can be too expensive. The difficulty is made more severe especially when we know that ADMMs, being intrinsically first-order methods, are prone to require a large number of outer iterations to compute even a moderately accurate approximate solution. As a result, further decomposition of the variables in problem (1.1) for getting easier subproblems, if possible, should be incorporated when designing ADMM-type methods for solving it. Unfortunately, even if the functions f and g in problem (1.1) are separable with respect to each subspace, say, $\mathcal{X}_i$ and $\mathcal{Y}_j$, the naive extension of the classic ADMM to multi-block cases is not necessarily convergent [2]. How to address the aforementioned issues is the key reason why the algorithmic development, as well as the corresponding convergence analysis, of multi-block variants of the ADMM has been an important research topic in convex optimization.

Of course, it is not reasonable to expect finding a general algorithmic framework that can achieve sterling numerical performance on a wide variety of different classes of linearly constrained multi-block convex optimization problems. Thus, in this paper our focus is on model (1.1), which is already quite versatile, for the following two reasons. Firstly, this model is general enough to handle quite a large number of convex composite optimization models from both the core convex optimization and realistic applications [4, 19]. Secondly, the convergence of multi-block variants of the ADMM for solving problem (1.1) has been separately realized in [4, 5, 18, 19, 25, 30], without sacrificing the numerical performance when compared to the naively extended multi-block ADMM. The latter has long been served as a benchmark for comparing new ADMM-type methods since its impressive numerical performance has been well recognized in extensive numerical experiments, despite its lack of theoretical convergence guarantee. Currently, this line of ADMMs has been applied to many concrete instances of problem (1.1), e.g., [1, 7, 11, 16, 21, 24, 26–29], to name just a few.