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Endpoint Estimates for Commutators of Fractional Integrals Associated to Operators with Heat Kernel Bounds

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Abstract: Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbf{R}^n)$ with pointwise upper bounds on heat kernel, and denote by $L^{-\alpha/2}$ the fractional integrals of L. For a BMO function $b(x)$, we show a weak type $L\log L$ estimate of the commutators $[b,\ L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x)$. We give applications to large classes of differential operators such as the Schrödinger operators and second-order elliptic operators of divergence form.

Key words: fractional integral, commutator, $L\log L$ estimate, semigroup, sharp maximal function

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1 Introduction and Main Results

Suppose that L is a linear operator on $L^2(\mathbf{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $a_t(x,\ y)$ satisfying an upper bound of the form

$$|a_t(x,\ y)| \leq t^{-\frac{n}{m}} g\left(\frac{|x-y|^m}{t}\right), \tag{1.1}$$

where m is a positive fixed constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r^m) = 0 \tag{1.2}$$

for some $\varepsilon > 0$.

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For $0 < \alpha < \dfrac{2n}{m}$, the fractional integrals $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\frac{\alpha}{2}} f(x) = \frac{1}{\Gamma\left(\dfrac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-tL}(f) \frac{\mathrm{d}t}{t^{-\frac{\alpha}{2}+1}}(x). \tag{1.3}$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbf{R}^n$, then $L^{-\frac{\alpha}{2}}$ is the classical fractional integrals $\mathcal{I}_\alpha$ (see, for example, Chapter 5 in [1]),

$$\mathcal{I}_\alpha(f)(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\frac{m\alpha}{2}}} \mathrm{d}y, \qquad 0 < \alpha < \frac{2n}{m}.$$

Let b be a BMO function on $\mathbf{R}^n$. The commutator of b and $L^{-\frac{\alpha}{2}}$ is defined by

$$[b,\ L^{-\frac{\alpha}{2}}](f)(x) = b(x)L^{-\frac{\alpha}{2}}(f)(x) - L^{-\frac{\alpha}{2}}(bf)(x).$$

It is well known that when $b \in \mathrm{BMO}(\mathbf{R}^n)$, the commutator $[b,\ \mathcal{I}_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, $1 < p < \dfrac{n}{\alpha}$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$ (see [2]), and of weak type $L\log L$ estimate for $p = 1$ (see [3] and [4]). For commutators of fractional integrals on homogeneous spaces, we refer the reader to [5], also to [6] for commutators of fractional integrals on non-homogeneous spaces.

The aim of this paper is to prove the following estimate.

Theorem 1.1 *Let $b \in BMO$, $\Phi(t) = t(1 + \log^+ t)$. Then for every $0 < \alpha < \dfrac{2n}{m}$, and $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m\alpha}{2n}$,*

(i) $\|[b,\ L^{-\frac{\alpha}{2}}]f\|_q \leq c\|b\|_*\|f\|_p$, $1 < p < \dfrac{2n}{m\alpha}$;

(ii) *When $p = 1$, $[b,\ L^{-\frac{\alpha}{2}}]$ is of weak type $L\log L$, that is,*

$$\begin{aligned} &|\{x \in \mathbf{R}^n : |[b,\ L^{-\frac{\alpha}{2}}](f)(x)| > \lambda\}|^{\frac{1}{q}} \\ &\leq C\left[\int_{\mathbf{R}^n} \Phi\left(\frac{\|b\|_*|f(x)|}{\lambda}\right)\mathrm{d}x\right]\left[1 + \frac{m\alpha}{2n}\log^+ \int_{\mathbf{R}^n} \Phi\left(\frac{\|b\|_*|f(x)|}{\lambda}\right)\mathrm{d}x\right], \end{aligned} \tag{1.4}$$

where $\|b\|_*$ denotes the BMO norm of $b(x)$.

Our result extends the results of [3] and [4] from $(-\Delta)$ to a general operator L, while we only assumes pointwise upper bounds on kernel $a_t(x,\ y)$ of e^{-tL} and no regularity on its space variables. Under our assumptions, the kernel of the operator $L^{-\frac{\alpha}{2}}$ does not have any regularity on space variables x and y. This allows flexibility on the choice of operator L in applications.

The paper is organized as follows. In Section 2, we recall some important estimates on BMO functions, maximal functions and fractional integrals. In Section 3, we prove some estimates on fractional integrals, which play a key role in the proof of the main result Theorem 1.1, which will be shown in Section 4 by using the approach of [4] and [7], combining with some estimates on the sharp maximal function $\mathcal{M}_L^{\#}f$. We conclude this paper by giving applications to large classes of differential operators which include the Schrödinger operators and second-order elliptic operators of divergence form.