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Bonnesen-style Isoperimetric Inequalities of an n-simplex

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Abstract: In this paper, by the theory of geometric inequalities, some new Bonnesen-style isoperimetric inequalities of n-dimensional simplex are proved. In several cases, these inequalities imply characterizations of regular simplex.

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1 Introduction

Let Ω_n be an n-simplex in the n-dimensional Euclidean space E^n with vertices A_1, A_2, $\cdots$, A_{n+1}. Denote by a_{ij} $(i, j = 1, 2, \cdots, n+1)$ the edge lengths of Ω_n (sometimes, we can set $a_1,\ a_2,\ \cdots,\ a_{\frac{1}{2}n(n+1)}$ in some order). If all edge lengths are equal, the simplex is said to be regular. Let F_i denote the $(n-1)$-dimensional volume of the facet $f_i = \{A_1,\ \cdots,\ A_{i-1},\ A_{i+1},\ \cdots,\ A_{n+1}\}$ opposite to the vertex P_i $(i = 1, 2, \cdots, n+1)$. Setting $F = \sum\limits_{i=1}^{n+1} F_i$, hence F is the surface area of Ω_n.

As a well known result, for a simple closed curve $\mathcal{C}$ (in the Euclidian plane) of length L enclosing a domain of area A, then

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

with equality holds if and only if the curve is a Euclidean circle. The quantity $L^2 - 4\pi A$ is said to be the isoperimetric deficit of $\mathcal{C}$ (see [1]–[3]).

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As an extension, Bonnesen proved the following inequality (see [1]):

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{1.2}$$

where R is the circumradius and r is the inradius of the curve $\mathcal{C}$. Note that if the right hand side of (1.2) equals zero, then $R = r$. This means that $\mathcal{C}$ is a circle and $L^2 - 4\pi A = 0$.

More generally, inequalities of the form

$$L^2 - 4\pi A \geq K \tag{1.3}$$

are called Bonnesen-style isoperimetric inequalities if equality is only attained for the Euclidean circle (see [1]). See references [4]–[9] for more details.

When the simple closed curve $\mathcal{C}$ is a triangle (in the Euclidean plane) of area S and with side lengths a_1, a_2, a_3, the following inequality is known:

$$P^2 \geq 3\sqrt{3} S, \tag{1.4}$$

where $P = \dfrac{1}{2}(a_1 + a_2 + a_3)$. Equality holds if and only if this triangle is regular.

Inequality (1.4) may be deemed isoperimetric inequality for triangles.

Veljan-Korchmaros inequality (see [10]) concerning the volume and the edge lengths of Ω_n states as follows:

$$\prod_{1 \leq i < j \leq n+1} a_{ij}^{\frac{2}{n+1}} \geq \left(\frac{2^n n!^2}{n+1} \right)^{\frac{1}{2}} V \tag{1.5}$$

with equality holds if and only if Ω_n is regular.

By utilize the arithmetic-geometric mean inequality to (1.5), we have

$$L^{2(n+1)} \geq \frac{n^{2(n+1)} (n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \tag{1.6}$$

with equality holds if and only if Ω_n is regular.

The inequality (1.6) may be deemed isoperimetric inequality of an n-simplex. The deficit value between the right-hand side and left-hand side of inequality (1.6) can be considered to be the isopermetric deficit for Ω_n:

$$\Delta_1 = L^{2(n+1)} - \frac{n^{2(n+1)} (n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}}. \tag{1.7}$$

In addition, the volume V and the facet areas of the simplex Ω_n satisfy the following inequality:

$$(V)^{\frac{2}{n}} \leq [(n-1)!]^{\frac{2}{n-1}} \frac{(n+1)^{\frac{1}{n}}}{n^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n+1} F_i \right)^{2(n^2-1)} \tag{1.8}$$

with equality holds if and only if Ω_n is regular (see [11]).

By applying the arithmetic-geometric mean inequality to (1.8), we have

$$F^{2(n^2-1)} \geq \left[\frac{n \cdot (n+1)^{\frac{1}{n}}}{(n-1)!^2} (n! \cdot V)^{\frac{2(n-1)}{n}} \right]^{n^2-1} \tag{1.9}$$

with equality holds if and only if Ω_n is regular.

The inequality (1.9) may be also called isoperimetric inequality for an n-simplex. The deficit value between the right-hand side and left-hand side of inequality (1.9) can be regarded as the other isopermetric deficit for the n-simplex Ω_n:

$$\Delta_2 = F^{2(n^2-1)} - \left[\frac{n \cdot (n+1)^{\frac{1}{n}}}{(n-1)!^2} (n! \cdot V)^{\frac{2(n-1)}{n}} \right]^{n^2-1}.$$