Journal of Computational Mathematics
Vol.38, No.1, 2020, 195–222.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1811-m2018-0162

# AN ERROR ANALYSIS METHOD SPP–BEAM AND A CONSTRUCTION GUIDELINE OF NONCONFORMING FINITE ELEMENTS FOR FOURTH ORDER ELLIPTIC PROBLEMS*

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**Abstract**

Under two hypotheses of nonconforming finite elements of fourth order elliptic problems, we present a side–patchwise projection based error analysis method (SPP–BEAM for short). Such a method is able to avoid both the regularity condition of exact solutions in the classical error analysis method and the complicated bubble function technique in the recent medius error analysis method. In addition, it is universal enough to admit generalizations. Then, we propose a sufficient condition for these hypotheses by imposing a set of in some sense necessary degrees of freedom of the shape function spaces. As an application, we use the theory to design a $P_3$ second order triangular $H^2$ non-conforming element by enriching two $P_4$ bubble functions and, another $P_4$ second order triangular $H^2$ nonconforming finite element, and a $P_3$ second order tetrahedral $H^2$ non-conforming element by enriching eight $P_4$ bubble functions, adding some more degrees of freedom.

*Mathematics subject classification:* 65N30, 73C02.
*Key words:* Nonconforming finite element, A priori error analysis, Biharmonic equation.

## 1. Introduction

We solve a biharmonic equation:

$$\begin{aligned} \Delta^2 u &= f, && \text{in } \Omega, \\ u = u_{\mathbf{n}} &:= \frac{\partial u}{\partial \mathbf{n}} = 0, && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded 2D polygonal domain or 3D polyhedral domain, and $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. Doing integration by parts twice, the weak formulation of (1.1) is: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \forall v \in H_0^2(\Omega). \tag{1.2}$$

Here $H_0^2(\Omega) := \{v \in H^2(\Omega) \mid v = v_{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$ and $H^2(\Omega)$ is the standard Sobolev space [3]. The bilinear forms are

$$\begin{aligned} a(u, v) &:= \int_\Omega D^2 u : D^2 v d\mathbf{x}, \\ (f, v) &:= \int_\Omega f v d\mathbf{x}, \end{aligned}$$

* Received July 30, 2018 / Revised version received September 13, 2018 / Accepted November 28, 2018 / Published online January 3, 2020 /

where

$$D^2u := \begin{pmatrix} \partial_{x_1x_1}u & \partial_{x_1x_2}u \\ \partial_{x_2x_1}u & \partial_{x_2x_2}u \end{pmatrix} \text{ for } 2D,$$

$$D^2u := \begin{pmatrix} \partial_{x_1x_1}u & \partial_{x_1x_2}u & \partial_{x_1x_3}u \\ \partial_{x_2x_1}u & \partial_{x_2x_2}u & \partial_{x_2x_3}u \\ \partial_{x_3x_1}u & \partial_{x_3x_2}u & \partial_{x_3x_3}u \end{pmatrix} \text{ for } 3D,$$

with $\partial_{x_1x_2}u := \frac{\partial^2 u}{\partial x_1 \partial x_2}$.

In traditional finite element methods of problem (1.2), degree 2 is the minimum polynomial degree. We need polynomials of degree 2 or above so that the second derivatives are not identically zero in the weak variational form. On a macro-triangle grid, the Powell-Sabin element [14] is a $P_2$ $H^2$-conforming finite element in 2D. That is, the finite element space is $C_1$, a subspace of the $H^2$ Sobolev space. The Hsieh-Clough-Tocher $P_3$ element is an $H^2$-conforming finite element on the 1-to-3 splitting macro-triangle grids, [6]. When the polynomial degree is 5 or above, single-triangle $H^2$-conforming elements can be constructed on general triangular grids [2]; when the polynomial degree is 9 or above, single-tetrahedron $H^2$-conforming elements can be constructed on general tetrahedral grids [21, 22]. On both non-macro-triangle grids and non-macro-tetrahedra grids, Morley element is a $P_2$ non-conforming finite element, i.e., the finite element space is not a subspace of the $H^2$ space. The remaining gap is the $P_3$ and $P_4$ non-conforming finite elements for the 4-th order differential equations in 2D and the $P_3, \cdots, P_8$ non-conforming finite elements for the 4-th order differential equations in 3D.

In this paper, we first present two hypotheses of nonconforming finite elements. Then, under them, we generalize the idea of [10, 12] to develop a side–patchwise projection based error analysis method (SPP–BEAM for short). Such a method only assumes the basic $H^2$ regularity for the exact solution. Compared with the classical a priori error analysis of nonconforming finite elements [3, 6, 16], the analysis herein applies integration by parts to discrete functions in nonconforming finite element spaces rather than the exact solution of the problem under consideration. This in particular allows to remove the indispensable regularity condition of the exact solution in the classical analysis. Compared with the recent medius analysis of [8, 10, 13], the analysis herein does not involve the bubble function technique which was first introduced to analyze efficiency of a posteriori error estimators [17]. Note that the bubble function technique will be very complicated for high dimensional cases and high order problems [8].

As an application of the theory, we propose a sufficient condition for these two hypotheses. More precisely, we give a set of in some sense necessary degrees of freedom of the possible shape functions space. Based on these degrees of freedom, we construct a $P_3$ second order triangular $H^2$ non-conforming finite element by enriching two $P_4$ bubble functions and imposing some additional degrees of freedom. As a result, the shape functions space is of 12 dimensions, and the corresponding degrees of freedom are the function value at three vertices and the average of function, the average of the normal derivative, and the first moment of the normal derivative, on three edges. Then, we construct another $P_4$, but still of 2nd order, $H^2$ non-conforming element on triangular grids. After that we design a $P_3$ second order tetrahedral $H^2$ non-conforming finite element by enriching eight $P_4$ bubble functions and adding some necessary degrees of freedom. The shape functions space of this three dimensional element is of 28 dimensions, and the corresponding degrees of freedom are the average and the first moment of function on six edges, and the average of function, the average of the normal derivative, and the first moment of the normal derivative, on four faces. We show that the three elements are well-defined