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HOW TO PROVE THE DISCRETE RELIABILITY FOR NONCONFORMING FINITE ELEMENT METHODS*

Carsten Carstensen and Sophie Puttkammer
Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
Email: cc@math.hu-berlin.de, puttkams@math.hu-berlin.de

Abstract

Optimal convergence rates of adaptive finite element methods are well understood in terms of the axioms of adaptivity. One key ingredient is the discrete reliability of a residual-based a posteriori error estimator, which controls the error of two discrete finite element solutions based on two nested triangulations. In the error analysis of nonconforming finite element methods, like the Crouzeix-Raviart or Morley finite element schemes, the difference of the piecewise derivatives of discontinuous approximations to the distributional gradients of global Sobolev functions plays a dominant role and is the object of this paper. The nonconforming interpolation operator, which comes natural with the definition of the aforementioned nonconforming finite element in the sense of Ciarlet, allows for stability and approximation properties that enable direct proofs of the reliability for the residual that monitors the equilibrium condition. The novel approach of this paper is the suggestion of a right-inverse of this interpolation operator in conforming piecewise polynomials to design a nonconforming approximation of a given coarse-grid approximation on a refined triangulation. The results of this paper allow for simple proofs of the discrete reliability in any space dimension and multiply connected domains on general shape-regular triangulations beyond newest-vertex bisection of simplices. Particular attention is on optimal constants in some standard discrete estimates listed in the appendices.

Mathematics subject classification: 65N30.
Key words: Discrete reliability, Nonconforming finite element method, Conforming companion, Morley, Crouzeix-Raviart, Explicit constants, Axioms of adaptivity.

1. Introduction

1.1. Motivation

The nonconforming finite element schemes are a subtile but important part of the finite element practice not exclusively in computational fluid dynamics [1–3], but also with benefits for guaranteed lower bounds of eigenvalues [5, 9], lower bounds for energies e.g. in the obstacle problem [13], or guaranteed convergence for a convex energy density despite the presence of the Lavrentiev phenomenon [23]. Prominent examples are Crouzeix-Raviart [17] and Morley [22] finite elements illustrated in Fig. 1.1.a and d.

The discrete reliability is one key-property in the overall analysis of optimal convergence rates in adaptive mesh-refining algorithms and one axiom in [4, 15]. Its proof is a challenge in the nonconforming setting since even given an admissible refinement $\widehat{\mathcal{T}}$ of an regular triangulation $\mathcal{T}$ the associated finite element spaces are non-nested $V(\widehat{\mathcal{T}}) \not\subset V(\mathcal{T})$.

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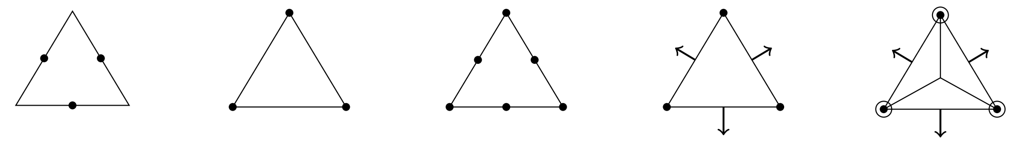

(a) Crouzeix-Raviart (b) P_1 (c) P_2 (d) Morley (e) HCT

Fig. 1.1. Mnemonic diagrams of the finite elements in 2D.

1.2. Methodology

The authors see three different arguments (i)–(iii) to circumvent the non-nestedness of the nonconforming schemes in the literature,

(i) appropriate mesh-refining,

(ii) discrete Helmholtz decomposition,

(iii) conforming companions.

For Crouzeix-Raviart finite elements see Theorem 2.1 in [24] for (ii). The restriction to simply-connected domains and dimension $n = 2$ from (ii) is circumvented in [7] for Crouzeix-Raviart using intermediate triangulations (i) and an associated discrete quasi-interpolation. For the Morley finite element analysis see Lemma 5.5 in [20] for (i) and Theorem 4.1 in [6] for (ii). This paper presents (iii) and its application for more general and refined results to prove discrete reliability. This general domain independent principle shall serve as a guideline for the many nonconforming methods in the rich literature. Often a discrete Helmholtz decomposition is not available, however the construction of a conforming companion although allows to compute guaranteed upper error bounds. Therefore, it seems intuitive to use this operator for the proof of discrete reliability as outlined in this paper.

1.3. Model Problems

For better intuition the reader may have the following model problems in mind. Given a polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a right-hand side $f \in L^2(\Omega)$, for a second-order problem consider the Poisson Model Problem, find $u \in H^1(\Omega)$ with

$$\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ along } \partial\Omega,$$

where the weak formulation seeks $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega).$$

The discrete version of this energy scalar product reads

$$a_h(u_h, v_h) := \int_\Omega \nabla_{\mathrm{NC}} u_h \cdot \nabla_{\mathrm{NC}} v_h \,\mathrm{d}x \quad \text{for all } u_h, v_h \in H^1(\Omega) + V(\mathcal{T}) + V(\widehat{\mathcal{T}}), \tag{1.1}$$

where a possible choice for the nonconforming finite element space $V(\mathcal{T})$ is the Crouzeix-Raviart space $CR^1_0(\mathcal{T})$. A simple fourth-order elliptic problem is the biharmonic equation, which seeks $u \in H^2(\Omega)$ with

$$\Delta^2 u = f \text{ in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ along } \partial\Omega.$$