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SUPERCONVERGENCE ANALYSIS OF THE POLYNOMIAL PRESERVING RECOVERY FOR ELLIPTIC PROBLEMS WITH ROBIN BOUNDARY CONDITIONS*

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Abstract

We analyze the superconvergence property of the linear finite element method based on the polynomial preserving recovery (PPR) for Robin boundary elliptic problems on triangulartions. First, we improve the convergence rate between the finite element solution and the linear interpolation under the H^1-norm by introducing a class of meshes satisfying the *Condition* (α, σ, μ). Then we prove the superconvergence of the recovered gradients post-processed by PPR and define an asymptotically exact a posteriori error estimator. Finally, numerical tests are provided to verify the theoretical findings.

Mathematics subject classification: 65N12, 65N15, 65N30.
Key words: Superconvergence, Polynomial preserving recovery, Finite element methods, Robin boundary condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygon with boundary $\Gamma := \partial\Omega$. Let $\mathbf{n}$ be the unit normal vector to the boundary exterior to Ω. We consider the supercenvergence analysis for the model problem: Find $u \in H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v + cuv) + \int_{\partial\Omega} quv = f(v) + g(v), \qquad \forall v \in H^1(\Omega), \tag{1.1}$$

where $c \in L^\infty$, $q \in L^\infty(\Gamma)$, $f \in H^{-1}(\Omega)$ and $g \in H^{-\frac{1}{2}}(\partial\Omega)$. We note that most results hold for a general class of elliptic equations and (1.1) is for presenting the main idea and techniques in their simplest form.

For given a shape regular triangulation $\mathcal{M}_h$ of $\bar{\Omega}$ with mesh size h, we denote

$$V_h := \Big\{ v_h \in H^1(\Omega) : v_h|_\tau \in P_1(\tau) \ \forall \tau \in \mathcal{M}_h \Big\}$$

the space of all continuous, piecewise linear finite element functions corresponding to $\mathcal{M}_h$. Here P_1 denotes the set of polynomials with degree at most one. The finite element solution $u_h \in V_h$ satisfies

$$a(u_h, v_h) = f(v_h) + g(v_h), \qquad \forall v \in H^1(\Omega). \tag{1.2}$$

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It is well known that there are many superclose and superconvergent results for Dirichlet boundary problems [13, 16, 18, 19, 22]. The convergence analysis is for uniform grids or patch symmetric grids at first. However, since it is difficult to construct uniform grids on unstructured domains and the grids produced by grid generation algorithms are a small perturbation of uniform grids in the most region of the domain, one considered the so-called mildly structured grids where an $O(h^{1+\alpha})$ approximate parallelogram property is satisfied for pairs of adjacent triangles in most parts of Ω except for a region of size $O(h^{2\sigma})$ [6, 7, 16, 18]. Two finite element functions vanishing on $\partial\Omega$, the continuous linear finite element solution u_h^0 and the continuous linear nodal interpolation u_I^0 of u^0, are superclose in the sense that

$$\|\nabla u_h^0 - \nabla u_I^0\|_{H^1(\Omega)} = O(h^{1+\min(\alpha,1-\sigma)}).$$

Here we assume that u^0 is the exact solution to the Dirichlet boundary problem. Based on the supercloseness, various post-processing techniques, such as the global L^2 projection [6, 8, 11], the Zienkiewicz-Zhu (ZZ) method [24, 25], and the Polynomial Preserving Recovery [13, 14, 23], have been proposed to produce a new approximation $R_h(u_h^0)$ of ∇u^0, which is superconvergent in the sense that

$$\|R_h(u_h^0) - \nabla u^0\|_{H^1(\Omega)} = O(h^{1+\min(\alpha,1-\sigma)}).$$

Based on the superconvergence results, an asymptotically exact error estimator can be constructed [7, 16]. In the last decade the convergence proof for Dirichlet boundary problem has been well established. By contrast, there are only a few superconvergent works on the Robin boundary problem. [9] considered the Robin boundary condition and proved the superconvergent rate of $O(h^{3/2})$. [3] considered the case of Neumann boundary and $\alpha = 1$ (i.e. each of the "good" pairs of triangles forms an $O(h^2)$ approximate parallelograms) and proved the superconvergent rate of $O(h^{2-\sigma}|\log h|^{\frac{1}{2}})$.

In this work, we investigate the superconvergence property of the method (1.2) when being post-processed by the polynomial preserving recovery (PPR) for the Robin boundary problem. PPR was proposed by Zhang and Naga [23] in 2004 and has been successfully applied to finite element methods. COMSOL Multiphysics adopted PPR as a post-processing tool since 2008, see [1]. One important feature of PPR is its superconvergence property for the recovered gradient. To learn more about PPR, readers are referred to [13, 16, 20, 21]. Some theoretical results about recovery techniques and recovery-type error estimators can be found in [4, 12, 18, 19, 22].

We first extend the definition of mildly structured grids to the boundary by assuming that the two triangles associated to a "good" boundary node are $O(h^{1+\alpha})$ approximate congruent triangles and the number of "bad" boundary nodes is of order $O(h^{-2\mu})$ for some $0 \le \mu < \frac{1}{2}$. Secondly, we prove the following supercloseness result:

$$\|u_h - u_I\|_{H^1(\Omega)} = O\Big(h^{1+\min(\alpha,1-\sigma)} + \min\left(h^{2-2\mu}\,|\log h|^{\frac{1}{2}}, h^{\frac{3}{2}}\right)\Big),$$

which improves the estimates of [3, 9]. Denote $G_h : V_h \to V_h \times V_h$ as the gradient recovery operator from PPR. Thirdly, we obtain the following estimate:

$$\|\nabla u - G_h u_h\|_{L^2(\Omega)} \lesssim h^{1+\min\{\alpha,1-\sigma\}} + \min(h^{2-2\mu}\,|\log h|^{\frac{1}{2}}, h^{3/2}). \tag{1.3}$$

Based on the superconvergent result, we define an asymptotically exact a posteriori error estimator $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$. Readers are referred to [2, 5, 10, 15] for further theoretical results about recovery techniques and recovery-type error estimators.