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THE QUADRATIC SPECHT TRIANGLE*

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Abstract

We propose a class of 12 degrees of freedom triangular plate bending elements with quadratic rate of convergence. They may be viewed as the second order Specht triangle, while the Specht triangle is one of the best first order plate bending element. The convergence result is proved under minimal smoothness assumption on the solution. Numerical results for both the smooth solution and nonsmmoth solution confirm the theoretical prediction.

Mathematics subject classification: 65N38, 65N30.
Key words: Specht triangle, Plate bending element, Basis functions

1. Introduction

Numerical approximation of the plate bending problem as well as other fourth order elliptic boundary value problems usually demand certain special devices. C^1 finite elements are required for the conforming finite elements [1, 2], which can be quite complicated, in particularly in three dimension. This stimulates the develop of the nonconforming finite elements. The Specht triangle [3] is a successful plate bending element, which passes all the patch tests and performs excellently, and is one of the best thin plate triangles with 9 degrees of freedom that currently available [2, citation in p. 345]. The Specht triangle employs quadratic polynomial approximation and hence is a first order plate bending element, which likes many practical nonconforming plate bending elements such as Zienkwiecz triangle [4], Morley triangle [5], just name a few of them. There are some second order nonconforming plate bending elements scattered in the literature, such as the one proposed by one of the author in [6] from the notion of the double set parameter method and the one in [7], both elements have 12 degrees of freedom. Other quadratic plate bending elements, such as those in [8] and [9], have 16 degrees of freedom. When we consider the rectangular mesh, the second order plate bending elements include the famous Adini element [10], the one proposed by one of the author in [11], and the one proposed in [12], in which a family of rectangular plate bending element is constructed. Compared to the first order plate bending elements, the choice for the second order plate bending elements is quite limited. It is worthwhile to mention that there are some higher order finite elements for

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the biharmonic problem in the framework of mixed finite elements and discontinuous Galerkin method; see; e.g., [13–16].

Motivated by the bubble function method, we propose a family of second order plate bending elements with 12 degrees of freedom, which could be regarded as the second order Specht triangle. The original motivation for the bubble function method is to design the stable finite element pair for the Stokes problem [17]. The basic idea of this method is to augment the finite element space by a bubble function space. The augmented bubble function space helps out in dealing with the extra constraints such as the divergence stability in Stokes problem and the high order consistency error. Besides being widely used in design stable finite elements in Stokes problem, the bubble function method has also been used to design efficient mass lumping method [18], to design robust finite elements for a singularly perturbed fourth order problem [8, 19], and it has been exploited by the authors to design robust finite elements for the strain gradient elasticity model [20]. In the context of the plate bending elements, certain classical elements such as the Zienkiewicz triangle and the Specht triangle can be derived by the bubble function method.

In the present work, the bubble function method is exploited to improve both the approximating error and the consistency error to the second order, which naturally yields a class of second order plate bending elements, which recovers the element in [7] as a special case. These elements are C^0 continuous. Therefore, they may be used to approximate the singular perturbation problem of fourth order as shown in [19], and to be exploited to construct robust strain gradient element as shown in [20, 21]. Based on the enriching operator technique in the discontinuous Galerkin method [22–24], we prove the convergence of the proposed elements under minimal smoothness assumption of the solution. Optimal rate of convergence is derived for solution in various Sobolev norms and broken norms. We also derived the optimal rate of convergence for the problem with Dirac-delta source term, which is particularly important for plate bending problem, because it corresponds to an idealization of a point load [25]. Numerical results for both the smooth solution and the nonsmooth solution support the theoretical prediction.

The structure of the paper is as follows. In the next section, we introduce the nonconforming finite element approximation of the plate bending problem. Detailed derivation of the new elements is presented in §3. The error estimates under minimal smoothness assumption are proved in §4. The numerical results for both the smooth solution as well as the nonsmooth solution are reported in the last section.

Throughout this paper, the constant C may differ at different occurrence, while it is independent of the mesh size h.

2. Finite Element Approximation of the Plate Bending Problem

To introduce the plate bending problem, we introduce some notations. The space $L^2(\Omega)$ of the square-integrable functions defined on a bounded polygon Ω is equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$. Let $H^m(\Omega)$ be the standard Sobolev space [26] with the norm and seminnorm defined as

$$\|v\|^2_{H^m(\Omega)}=\sum_{k=0}^{m}|v|^2_{H^k(\Omega)}\quad\text{and}\quad|v|^2_{H^k(\Omega)}=\int_\Omega\sum_{|\alpha|=k}|\nabla^\alpha v|^2\mathrm{d}x,$$