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VARIATIONAL DISCRETIZATION OF A CONTROL-CONSTRAINED PARABOLIC BANG-BANG OPTIMAL CONTROL PROBLEM*

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Abstract

We consider a control-constrained parabolic optimal control problem without Tikhonov term in the tracking functional. For the numerical treatment, we use variational discretization of its Tikhonov regularization: For the state and the adjoint equation, we apply Petrov-Galerkin schemes in time and usual conforming finite elements in space. We prove a-priori estimates for the error between the discretized regularized problem and the limit problem. Since these estimates are not robust if the regularization parameter tends to zero, we establish robust estimates, which — depending on the problem's regularity — enhance the previous ones. In the special case of bang-bang solutions, these estimates are further improved. A numerical example confirms our analytical findings.

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Key words: Optimal control, Heat equation, Control constraints, Finite elements, A-priori error estimates, Bang-bang controls.

1. Introduction

In this article we are interested in the numerical solution of the optimal control problem

$$\min_{u\in U_{\mathrm{ad}}} J_0(u) \quad \text{with} \quad J_0(u) := \frac{1}{2}\|Tu - z\|_H^2. \tag{$\mathbb{P}_0$}$$

Here, T is basically the (weak) solution operator of the heat equation, the set of admissible controls U_{ad} is given by box constraints, and $z \in H$ is a given function to be tracked.

Often, the solutions of ($\mathbb{P}_0$) possess a special structure: They take values only on the bounds of the admissible set U_{ad} and are therefore called *bang-bang solutions*.

Theoretical and numerical questions related to this control problem attracted much interest in recent years, see, e.g., [1–11]. The last four papers are concerned with T being the solution operator of an *ordinary* differential equation, the former papers with T being a solution operator of an *elliptic* PDE or T being a continuous linear operator. In [12], a brief survey of the content of these and some other related papers is given at the end of the bibliography.

Problem ($\mathbb{P}_0$) is in general ill-posed, meaning that a solution does not depend continuously on the datum z, see [3, p. 1130]. The numerical treatment of a discretized version of ($\mathbb{P}_0$) is also challenging, e.g., due to the absense of formula (2.10) in the case $\alpha = 0$, which corresponds to problem ($\mathbb{P}_0$).

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Therefore we use Tikhonov regularization to overcome these difficulties. The *regularized problem* is given by

$$\min_{u\in U_{\mathrm{ad}}} J_\alpha(u) \quad \text{with} \quad J_\alpha(u) := \frac{1}{2}\|Tu - z\|_H^2 + \frac{\alpha}{2}\|u\|_U^2 \tag{$\mathbb{P}_\alpha$}$$

where $\alpha > 0$ denotes the regularization parameter. Note that for $\alpha = 0$, problem $(\mathbb{P}_\alpha)$ reduces to problem $(\mathbb{P}_0)$.

For the numerical treatment of the regularized problem, we then use variational discretization introduced by Hinze in [13], see also [14, Chapter 3.2.5]. The state equation is treated with a Petrov-Galerkin scheme in time using a piecewise constant Ansatz for the state and piecewise linear, continuous test functions. This results in variants of the Crank-Nicolson scheme for the discretization of the state and the adjoint state, which were proposed recently in [15]. In space, usual conforming finite elements are taken. See [12] for the fully discrete case and [16] for an alternative discontinuous Galerkin approach.

The purpose of this paper is to prove a-priori bounds for the error between the discretized regularized problem and the limit problem, i.e. the continuous unregularized problem.

We first derive error estimates between the discretized regularized problem and its continuous counterpart. Together with Tikhonov error estimates recently obtained in [17], see also [12], one can establish estimates for the total error between the discretized regularized solution and the solution of the continous limit problem, i.e. $\alpha = 0$. Here, second order convergence in space is not achievable and (without coupling) the estimates are not robust if α tends to zero. Using refined arguments, we overcome both drawbacks. In the special case of bang-bang controls, we further improve those estimates.

The obtained estimates suggest a coupling rule for the parameters α (regularization parameter), k, and h (time and space discretization parameters, respectively) to obtain optimal convergence rates which we numerically observe.

The paper is organized as follows.

In the next section, we introduce the functional analytic description of the regularized problem. We recall several of its properties, such as existence of a unique solution for all $\alpha \geq 0$ (thus especially in the limit case $\alpha = 0$ we are interested in), an explicit characterization of the solution structure, and the function space regularity of the solution. We then introduce the Tikhonov regularization and recall some error estimates under suitable assumptions. In the special case of bang-bang controls, we recall a smoothness-decay lemma which later helps to improve the error estimates for the discretized problem.

The third section is devoted to the discretization of the optimal control problem. At first, the discretization of the state and adjoint equation is introduced and several error estimates needed in the later analysis are recalled. Then, the analysis of variational discretization of the optimal control problem is conducted.

The last section discusses a numerical example where we observe the predicted orders of convergence.

2. The Continuous Optimal Control Problem

2.1. Problem setting and basic properties

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a spatial domain which is assumed to be bounded and convex with a polygonal boundary $\partial\Omega$. Furthermore, a fixed time interval $I := (0, T) \subset \mathbb{R}$, $0 < T < \infty$,