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LOCAL PRESSURE CORRECTION FOR THE STOKES SYSTEM*

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Abstract

Pressure correction methods constitute the most widely used solvers for the time-dependent Navier-Stokes equations. There are several different pressure correction methods, where each time step usually consists in a predictor step for a non-divergence-free velocity, followed by a Poisson problem for the pressure (or pressure update), and a final velocity correction to obtain a divergence-free vector field. In some situations, the equations for the velocities are solved explicitly, so that the numerical most expensive step is the elliptic pressure problem. We here propose to solve this Poisson problem by a domain decomposition method which does not need any communication between the sub-regions. Hence, this system is perfectly adapted for parallel computation. We show under certain assumptions that this new scheme has the same order of convergence as the original pressure correction scheme (with global projection). Numerical examples for the Stokes system show the effectivity of this new pressure correction method. The convergence order $\mathcal{O}(k^2)$ for resulting velocity fields can be observed in the norm $l^2(0,T;L^2(\Omega))$.

Mathematics subject classification: 76D07, 65M55.
Key words: Stokes system, Navier-Stokes, Pressure correction, Finite elements.

1. Introduction

Pressure correction methods are the most widely used methods to solve the time-dependent Navier-Stokes equations, because they open the possibility to decouple the momentum equation from the divergence equation and, hence, lead to the possibility to use different solution techniques for obtaining the velocity and the pressure.

Since the pioneering work of Chorin and Temam [2, 8] many different pressure correction methods have been proposed. An extensive overview was provided by Guermond et al [3]. These methods commonly contain a prediction step for a not necessarily divergence-free velocity field, followed by a Poisson problem for the pressure (or a pressure update), and finally a projection of the previously computed velocity onto a divergence free one. In certain applications with the need of small time steps due to accuracy reasons, the predictor step can be formulated in an explicit way. Of course this requires an explicit treatment of the convective and diffusive term, so that severe time step restrictions are needed for stability. However, this is acceptable for typical applications as for instance in climate research. In this case, a numerically very expensive part of the splitting scheme is the Poisson problem for the pressure (update). It is a global problem with an associated matrix to be inverted with condition number dependent on the mesh resolution. For finer meshes, the condition number becomes larger, so that many iterative solvers (such as conjugate gradient methods, Jacobi iteration and Gauss-Seidel methods) suffer in terms of convergence rates. Parallelization on multi-core computers or parallel computers

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may help to reduce the simulation time but always require suitable further iteration techniques to account for the elliptic character of the Poisson problem.

Therefore, we propose in this work an alternative splitting method which replaces the global Poisson problem for the pressure update by a number of smaller non-overlapping Poisson problems that are completely decoupled. This gives the possibility to solve the pressure correction step in parallel without any communication within the iterative linear solver. In terms of accuracy, the resulting scheme is comparable to the original scheme. To a certain extent, the new method presented and analyzed here can be considered as an extension of the coarse grid projection method introduced by Lentine et al. [7] and recently studied by Kashefi and Staples [6].

We present this modified scheme, demonstrate several properties, and show first numerical results. The starting splitting scheme is not restricted to a special one. However, for ease of presentation, we consider in this work the pressure correction scheme of Timmermans et al. [9], which has been analyzed by Shen and Guermond [4] and is considered to be among the most accurate ones.

We start with a standard and well-analyzed pressure correction method in Section 2. The new local pressure correction scheme is introduced in Section 3, where we still give freedom of a concrete projection. The a priori analysis is topic of Section 4. To this end we formulate a necessary condition of the projection (Assumption 4.1). One possible realization of the local projection is given in Section 5. We also verify the assumption needed previously to prove the error estimate. First numerical results are subject of Section 6. We end with a short conclusion and outlook.

2. Pressure Correction for the Stokes System

We consider the time-dependent Stokes equations in a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, homogeneous Dirichlet conditions on $\partial\Omega$ and initial velocity field $\boldsymbol{u}_0$. With the velocity field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, the pressure $p : \Omega \to \mathbb{R}$, and a forcing term $\mathbf{f} : \Omega \to \mathbb{R}^d$ the Stokes system reads

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega_T := \Omega \times (0,T], \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega_T, \tag{2.2}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T], \tag{2.3}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{2.4}$$

Here, $\nu > 0$ is a positive, constant viscosity. We take $M+1$ discrete time points $0 = t_0, t_1, \ldots, t_M$ with for simplicity a constant time step $k = t_m - t_{m-1} > 0$ for all m, hence $t_m := mk$ for $0 \leq m \leq M = T/k$. The outer normal on the boundary $\partial\Omega$ is denoted by $\boldsymbol{n}$. The so-called rotational incremental pressure-correction proposed by Timmermans et al. [9] starts with a given $\boldsymbol{u}_0$ and p_0 and reads for $m \geq 1$:

Step 1: The velocity predictor $\tilde{\boldsymbol{u}}_m$ is obtained by solving the momentum equation with a BDF(2)-scheme and a given pressure p_{m-1}:

$$\frac{1}{2k}(3\tilde{\boldsymbol{u}}_m - 4\boldsymbol{u}_{m-1} + \boldsymbol{u}_{m-2}) - \nu\Delta\tilde{\boldsymbol{u}}_m = \mathbf{f}(t_m) - \nabla p_{m-1} \qquad \text{in } \Omega, \tag{2.5}$$

$$\tilde{\boldsymbol{u}}_m = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{2.6}$$