Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 2 (2019), pp. 144-162

DOI: 10.4208/ata.OA-0003

The Neumann Problem of Complex Special Lagrangian Equations with Supercritical Phase

Chuanqiang Chen[1], Xinan Ma[2,*] and Wei Wei[2]

[1] *Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, 310023, Zhejiang, China*
[2] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, Anhui, China*

Received 22 August 2017; Accepted (in revised version) 8 June 2018

Abstract. Inspired by the Neumann problem of real special Lagrangian equations with supercritical phase, we consider the Neumann problem of complex special Lagrangian equations with supercritical phase in this paper, and establish the global C^2 estimates and the existence theorem by the method of continuity.

Key Words: Special Lagrangian equation, Neumann problem, supercritical phase.

AMS Subject Classifications: 35J60, 35B45

1 Introduction

As we all know, the real special Lagrangian equation is

$$\arctan D^2 u = \Theta(x), \tag{1.1}$$

where

$$\arctan D^2 u := \arctan\lambda_1 + \arctan\lambda_2 + \cdots + \arctan\lambda_n,$$

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix

$$D^2 u = \Big\{ \frac{\partial^2 u}{\partial x_i \partial x_j} \Big\}_{1\le i,j\le n},$$

and Θ is called the phase. In particular, $\Theta = \frac{(n-2)\pi}{2}$ is the critical phase, and if $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$, Eq. (1.1) is called special Lagrangian equations with supercritical phase.

*Corresponding author. *Email addresses:* chuanqiangchen@zjut.edu.cn (C. Q. Chen), xinan@ustc.edu.cn (X. N. Ma), weisx001@mail.ustc.edu.cn (W. Wei)

The special Lagrangian equation (1.1) was introduced by Harvey-Lawson [13] in the study of calibrated geometries. Here Θ is a constant called the phase angle. In this case the graph $x\mapsto(x,Du(x))$ defines a calibrated, minimal submanifold of $\mathbb{R}^{2n}$. Since the work of Harvey-Lawson, special Lagrangian manifolds have gained wide interest, due in large part to their fundamental role in the Strominger-Yau-Zaslow description of mirror symmetry [26].

For the special Lagrangian equations with supercritical phase, Yuan obtained the interior C^1 estimate with Warren in [29] and the interior C^2 estimate with Wang in [28]. Recently Collins-Picard-Wu [8] obtained the existence theorem of the Dirichlet problem.

Moreover, for the Dirichlet problem of elliptic equations in $\mathbb{R}^n$, many results are known. For example, the Dirichlet problem of Laplace equation is studied in [7, 11], Caffarelli-Nirenberg-Spruck [2] and Ivochkina [15] solved the Dirichlet problem of Monge-Ampère equation, and Caffarelli-Nirenberg-Spruck [4] solved the Dirichlet problem of k-Hessian equation. After the pioneering works of Caffarelli et al., the Dirichlet problem of the general Hessian quotient equation was solved by Trudinger in [27]. For more information about the related subjects, we refer to the citations of [2, 4, 15].

Also, the Neumann or oblique derivative problem of partial differential equations was widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book [11]. Also, we can see the recent book written by Lieberman [21] for the Neumann and the oblique derivative problems of linear and quasilinear elliptic equations. In 1986, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equation in the celebrated paper [23]. Recently, Ma-Qiu [24] solved the the Neumann problem of k-Hessian equations, and Chen-Zhang [6] generalized the result to the the Neumann problem of Hessian quotient equations. For the Neumann problem of special Lagrangian equations with supercritical phase, Chen-Ma-Wei [5] got the existence theorem. In [16, 17], Jiang-Trudinger studied the general oblique boundary value problems for augmented Hessian equations with some regular condition and some concavity condition.

At the same time, the complex equations have attracted a variety of mathematicians and many excellent works have been done. The complex Monge-Ampère equations are definitely one of the most important equations in partial differential equation and the geometry. In [1], Bedford and Taylor studied the Dirichlet problem of complex Monge-Ampère equations by using the Perron-Bremermann family method, and got the existence and uniqueness of the weak solutions and a global Lipschitz regularity for plurisubharmonic solution when Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. In [3], Caffarelli et al. studied the classical solution on strongly pseudoconvex domains. Their work was extended to arbitrary bounded domains in $\mathbb{C}^n$ by Guan in [12] under some subsolution condition. Recently, Fu-Yau equation on compact Kähler manifolds arises much attention, which was introduced in [10]. The Fu-Yau equation was solved in dimension 2 by Fu-Yau and was recently extended to higher dimensions in some cases in [25] by Phong, Picard and Zhang. In high dimensions, the equation is actually a 2-Hessian type equation. Complex Hessian equations have been studied extensively by many authors